The University of Alberta
Printing Department
Edmonton, Alberta

INTUITIVE THINKING

(C)

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF EDUCATION

DEPARTMENT OF SECONDARY EDUCATION

THE UNIVERSITY OF ALBERTA

INTUITIVE THINKING IN THE MATHEMATICS CLASSROOM

by

© Edward Klatt

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF EDUCATION

DEPARTMENT OF SECONDARY EDUCATION

EDMONTON, ALBERTA

July, 1968

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Intuitive Thinking in the Mathematics Classroom," submitted by Edward Klatt in partial fulfilment of the requirements for the degree of Master of Education.

ABSTRACT

The purpose of the study was to develop an instrument for categorizing intuitive thinking and for rank-ordering teachers as to the extent to which they make use of an intuitive approach in teaching mathematics. Particular attention was given to establishing a basis in learning theory for the role of intuition and in giving a definition of intuitive thinking in terms of observable behavior.

A theoretical analysis discusses the complementary nature of analytical and intuitive thinking as expounded by Bruner, Evans, Dienes, Polya and others. The analysis resulted in ten "elements of intuitive thinking:" symmetry, analogy, similarity, generalization, experimentation, manipulation, skipping steps, preverbalization, solution visualization, and informal proof. Intuitive thinking was defined to be thinking which makes use of the above elements.

Dienes' theory of mathematics learning was used to explain the function of intuitive thinking. The point of view presented is that intuitive thinking can give the learner mathematical imagery, the lack of which is considered to be an important contributing factor to difficulties in understanding mathematics.

The review of related research included several projects on instrument development. From these earlier studies, a procedure for instrument development was formulated which

then served as the basic design for the present study. Several studies on intuition were also reviewed. These illustrated that the concept has and can be studied quantitatively.

For developing the instrument, transcribed classroom lessons were used. The transcriptions were first broken down into units and then each unit was classified as being one of the ten elements of intuitive thinking or as being non-intuitive.

The instrument was then used to analyze several lessons. Both in this analysis and in the development of the instrument there was a general scarcity of intuitive thinking, while "preverbalization" was completely absent and "skipping steps" was used only once.

Since this is an exploratory study dealing with a subject seldom studied empirically, several significant implications for future research are given. A conclusion of the study was that the instrument needs to be further evaluated by using it in whole or part to determine its usefulness as a means of providing teacher feedback or as a means for doing research.

ACKNOWLEDGMENTS

I am grateful to the many persons who assisted me in completing this study.

In particular I wish to express my appreciation to the following:

Dr. S. E. Sigurdson, my supervisor, whose guidance, encouragement, and interest helped me to formulate the subject of the thesis and see it to its conclusion;

The members of my committee, whose attention to style and expression increased the quality of this report;

The Edmonton Public School Board and the principals and teachers in the schools in which the lessons were recorded, without whose cooperation the study could not have been completed;

My fellow Graduate Students, whose interest and comments during formal and informal discussions aided greatly in clarifying ideas;

My wife Lora, whose extensive assistance in developing criteria for the instrument and classifying units contributed directly to the development of the instrument, and whose patient love and encouragement through the months enabled me to apply myself to the work of this study.

E. K.

TABLE OF CONTENTS

CHAPTER		PAGE
I.	THE NATURE OF THE PROBLEM	1
	Introduction	1
	The Need for the Study	1
	The Purpose of the Study	2
	Delimitations	3
	Significance of the Study	4
	Definition of Terms	5
II.	THE THEORETICAL FRAMEWORK	6
	Introduction	6
	Analytic Thinking	6
	Intuitive Thinking	8
	Definitions	8
	Mathematical creativity	11
	Learning efficiency	14
	Elements of intuitive thinking	16
	Complementary Nature of Intuitive and Analytical Thinking	19
	Developmental Aspects of Intuitive and Analytical Thinking	20
	Chronological development of mental capacities	21
	Concept development	22
	A Behavioral Definition of Intuitive Thinking	23

CHAPTER PAGE

III. A THEORY OF MATHEMATICS LEARNING AND
INTUITIVE THINKING 33
Introduction 33
Mathematics and Mathematics Learning 33
Background Information on Contemporary
Learning Theories 35
Dienes' Theory of Mathematics Learning 38
The Function of Intuitive Thinking 42

IV. A REVIEW OF SELECTED RESEARCH 46
Introduction 46
Measurement of Classroom Behavior
By Systematic Observation 47
The B. O. Smith study 47
The Wright-Proctor study 51
Measurement of Intuitive Thinking 53
Chapter Summary 55

V. THE DESIGN OF THE STUDY 56
Introduction 56
Description of the Data 56
The Procedure for Developing the Instrument . . 57
The unit 59
The categories 60
Analysis of Classroom Discourse Using the
Instrument 62

CHAPTER PAGE

VI. THE RESULTS OF THE STUDY 63
Introduction 63
Requirements of the Units 63
Choice of the Unit 65
Unit Identification Criteria 69
Reliability of Unit Identification Criteria . . 73
Requirements of the Categories 74
Developing the Categories 75
The Part of the Unit to be Used in
Classifying Units 75
Unit Classification Criteria 76
Reliability of the Unit Classification
Criteria 81
Results of the Classroom Analysis 83
VII. CONCLUSIONS AND IMPLICATIONS 85
Introduction 85
Conclusions 85
General Observations 86
Some Problems with the Present Instrument . . . 88
Implications for Teachers 89
Implications for Future Research 90
BIBLIOGRAPHY 93
APPENDIX 100

LIST OF TABLES

TABLE		PAGE
I.	Reliability of the Unit Identification Criteria . .	73
II.	Reliability of Unit Classification Criteria	82
III.	Classification of the Units of the B Series Lessons by Categories	83
IV.	Intuitive Ratio by Classroom	83

CHAPTER I

THE NATURE OF THE PROBLEM

I. INTRODUCTION

In the last one and a half decades, much research has centered on developing suitable mathematics curricula for schools. We seem to be much closer to this goal than we are to the goal of providing adequate instruction in these courses. It is clear that the "curriculum planned" and the "curriculum had" are not the same thing. It is the method of instruction employed by the teacher and the student's response to the method employed that ultimately determines the "curriculum had" for each individual student. This study is concerned with methods of instruction and, in particular, with the intuitive approach.

II. THE NEED FOR THE STUDY

The motivation for this study stems from Bruner's plea:

> Unfortunately, the formalism of school learning has somehow devalued intuition. It is the very strong conviction of men who have been designing curricula, in mathematics and science particularily, that much work is needed to discover how we may develop the intuitive gifts of our students from the earliest grades onward.[1]

---

[1] Bruner, Jerome S., _The Process of Education_ (New York: Vintage Books, 1960), pp. 59-60.

We might first of all ask for the empirical evidence that intuition is devalued. An attempt to answer this question could be made if one had an instrument to analyze classroom discourse one of whose dimensions would give a measure along the intuitive-analytic continuum. Then we might be able to question the value of the intuitive approach. Some kind of instrument which would rate teachers' instructional modes as being more intuitive or less intuitive could be useful in correlating the teacher's approaches with stated student outcomes. Is the development of intuitive thinking in students more likely if their teachers think intuitively? An instrument which would give a measure of the extent to which teachers vary in using the intuitive approach would go a long way to answering the question.[2] In Bruner's words:

> It is certainly clear that procedures or instruments are needed to characterize and measure intuitive thinking, and that the development of such instruments should be pursued vigorously.[3]

III. THE PURPOSE OF THE STUDY

The purpose of this study is to develop an instrument for categorizing intuitive thinking and for rank-ordering classroom sessions on the basis of the extent to which they make use of intuitive thinking in a mathematics teaching-

---

[2]The other part needed, of course, is an instrument to measure the extent to which students think intuitively.

[3]Bruner, The Process of Education, p. 61.

learning situation. A secondary purpose is to illustrate the use of the instrument so constructed.

The study thus attempts to give a partial answer to Bruner's question:

> We may ask if we can learn to agree in classifying a person's style or preferred mode of working as characteristically more analytic or inductive on the one hand, or more intuitive?[4]

The answer to Bruner's question attempted in this study is incomplete in at least two ways. In the first place, this study considers only audible verbal discourse in the classroom, and thereby attempts to classify the "classroom's" characteristic mode of thinking. This is, of course, largely determined by the teacher. In the second place, the study focuses only on characteristics of intuitive thinking.

IV. DELIMITATIONS

The instrument devised is not intended to give an absolute measure of the "amount" of intuitive thinking engaged in by members of the classroom. The purpose of the instrument is only to point out <u>variations</u> in the extent to which use is made of intuitive thinking.

The extent to which the instrument measures intuitive thinking is directly dependent on the definition, in terms of observable behaviour which is the starting point. Hence

---

[4] *Ibid.*, p. 59

the usefulness of the instrument for investigating intuitive processes will be limited by the extent to which the user agrees with the definition of intuitive thinking as set forth in this study.

Another limitation has to do with the very nature of intuitive thinking. One of the characteristics of intuitive thinking is the inability, oftentimes, of the intuiter to verbalize his approach to a solution. Indeed, forcing him to verbalize may force him to think analytically. By limiting the instrument to analyzing verbal discourse we are, in fact, forced to limit the breadth of our definition of intuitive thinking.

V. SIGNIFICANCE OF THE STUDY

The study is significant in that it explores the much neglected, yet potentially productive field of the intuitive approach to learning. Such a study can hopefully provide significant guidance and stimulus to further research in this area.

More specifically, an acceptable instrument of the type developed here would be useful in determining the relationship between intuitive mathematical learning and student outcomes, and in establishing the extent to which teachers make use of an intuitive approach in the teaching of mathematics.

Thus, success in this project could open the doors to an extensive field of much needed research in mathematics

teaching methodology.

VI. DEFINITION OF TERMS

Classroom - refers to the pupils and teacher audibly participating in a teaching-learning situation.

The Instrument - is the device being developed in this study for the purpose of rank-ordering classrooms on the basis of the extent to which they make use of intuitive thinking during a mathematics teaching-learning situation.

Verbal Discourse - refers to the total audible verbal exchange among all members of a classroom during a normal school timetable period.

Unit of Verbal Discourse - is any subdivision of a verbal discourse, developed for the purpose of analyzing the verbal discourse.

Category - is an item on the instrument into which the various units of verbal discourse will be classified.

Unit Classification Criteria - are the criteria used for putting individual units of verbal discourse into particular categories.

Utterance - is the complete verbal expression of one individual before another takes over.

Elements of Intuitive Thinking - are the observable events which are taken to be manifestations of intuitive thinking.

CHAPTER II

THE THEORETICAL FRAMEWORK

I. INTRODUCTION

In mathematics, both analytic and intuitive thinking play important roles. Polya recognized these roles:

> Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science. Both aspects are as old as the science of mathematics itself.[1]

But words like analytic thinking and intuitive thinking are often used with different shades of meanings by different people. The purpose of this chapter is to examine the usage of these words in order to arrive at a behavioral definition of intuitive thinking which will be useful as a basis for constructing an instrument to measure variations in the extent to which use is made of intuitive thinking during a mathematics teaching-learning situation.

II. ANALYTIC THINKING

The creative process has often been described as occurring in four stages--preparation, incubation, illumination,

---

[1]Polya, G., How to Solve It (Garden City, New York: Doubleday Anchor Books, 1957), p. vii.

and verification.[2] Evans suggests that "in the verification stage, the analytic process is employed to verify or reject the proposed solution or discovery."[3]

Bruner has described the analytic process as follows:

> Analytic thinking characteristically proceeds a step at a time. Steps are explicit and usually can be adequately reported by the thinker to another individual. Such thinking proceeds with relatively full awareness of the information and operations involved. It may involve careful and deductive reasoning, often using mathematics or logic and an explicit plan of attack. Or it may involve a step-by-step process of induction and experiment, utilizing principles of research design and statistical analysis.[4]

Evans points out, however, that as used in a mathematical sense, the first of these two alternatives is the only acceptable one of verifying or rejecting a proposed solution or discovery.[5]

Polya characterizes analytic thinking as being safe, final, having rigid standards, being coded and clarified logic. It is, he says, "incapable of yielding essentially new knowledge about the world around us."[6]

---

[2]Evans, E.W., "Measuring the Ability of Students to Respond in Creative Mathematical Situations at the Late Elementary and Junior High School Level" (unpublished doctoral dissertation, The University of Michigan, 1964), p. 3.

[3]*Ibid*.

[4]Bruner, Jerome S., *The Process of Education* (New York: Vintage Books, 1960), p. 57.

[5]Evans, *op. cit.*, p. 21.

[6]Polya, G., *Mathematics and Plausible Reasoning* (Princeton, N.J.: Princeton University Press, 1954), I, p.v.

Analytic thinking does, of course, have a useful and important function. This function will be explained following our consideration of the intuitive process.

III. INTUITIVE THINKING

Referring to the four stages of the creative process named previously, Evans says that "intuition assumes the major role in the formulation of ideas and presentation of combinations (during the incubation stage) which eventually lead to the illumination stage."[7] After considering several definitions offered for intuitive thinking, our discussion will turn to the role of intuition referred to by Evans--its role in mathematical creativity and its role in increasing learning efficiency. Our attention will then be directed to what will be referred to here as "the elements of intuitive thinking."

Definitions. Bruner, who has a very high regard for the intuitive process, defines intuition as follows:

> For a working definition of intuition, we do well to begin with Webster: "immediate apprehension or cognition." Immediate in this context is contrasted with "meditated"--apprehension or cognition that depends on the intervention of formal methods of analysis and proof. Intuition implies the act of grasping the meaning, significance, or structure of a problem without explicit reliance on the analytic apparatus of one's craft.[8]

---

[7]Evans, op. cit., p. 20.

[8]Bruner, The Process of Education, p. 60.

He goes on to say that by using intuitive thinking, hypotheses are developed more quickly and long before their worth is known. It does not advance in well-defined steps and tends to involve maneuvers based on implicit perceptions of the problem. The intuiter is not aware of the process by which he reached a solution, nor is he aware of just what aspects of the problem he responded to.[9]

Ausubel, who assigns a more limited role to intuitive thinking, defines it in terms which are similar to those of Bruner's:

> Intuitive cognitive functioning refers to an implicit, relatively unprecise and informal type of understanding or thought process in which the individual is only vaguely aware of the parameters of a problem or its solution and of the logical operations involved.[10]

He characterizes the process with the following phrases: nonanalytic, unsystematic, relying on immediate apprehension, and defying explicit formulation of processes. He categorizes the intuitive process into four categories: developmental, unsophisticated, sophisticated and creative.[11]

Westcott has attempted empirical studies of intuition.[12] He says that the defining characteristic of the intuitive process "seems to be that an individual reaches his conclusion,

---

[9] Ibid.

[10] Ausubel, D.P., The Psychology of Meaningful Verbal Learning (New York: Grune and Stratton, 1963), p. 122.

[11] Ibid.

[12] These are reviewed in Chapter IV.

solution, or whatever, without being aware of how he reached it."[13] On the question of whether the correctness or incorrectness of the solution or the discovery defines an intuitive situation, Westcott feels it is not essential to differentiate.[14]

Sarbin, Taft and Baily have quoted Wild's definition of intuition from his classical work on intuition:

> An intuition is an immediate awareness by a subject of some particular entity without such aid from the senses or from reason as would account for that awareness.[15]

They state that there are four referents of intuition: (1) the achievement of an object or event (a cognition or knowing) (2) through a method which is inexplicable and unanalyzable (a mysterious process) (3) which carries the conviction of truth (certainty, complete credibility) and (4) which occurs without intervention of previously acquired cognitions (immediacy).[16]

Lovell puts the definition in a mathematical setting, though the process is similar to those defined above:

By mathematical intuition is meant the insight one

---

[13]Westcott, M.R., "Empirical Studies of Intuition," Widening Horizons of Creativity, C.W. Taylor, editor (New York: Wiley, 1964), p. 35.

[14]Ibid., p. 36.

[15]Sarbin, T.R., R. Taft and D.E. Bailey, Clinical Inference and Cognitive Theory (New York: Holt, Rinehart and Winston, 1960), p. 182.

[16]Ibid.

> gains, without the intervention of conscious reasoning, into numerical, algebraic, geometric and other mathematical phenomena.[17]

Getzels defines intuitive thinking as "thought not yet freed from perception."[18] Evans says that "intuitive thinking is the thought processes by which one gains insight in a learning situation, without the intervention of conscious deductive reasoning."[19]

In summary, all of the definitions either directly refer to, or imply the following defining characteristics of intuitive thinking:

1. A concept has been established or a solution has been found which is satisfying to the intuiter.
2. The logical steps leading to the result are unknown.
3. The result is immediate.

Mathematical Creativity. Brune,[20] Hohn,[21] Kemeny,[22]

---

[17]Lovell, K., The Growth of Basic Mathematical and Scientific Concepts in Children (London: University of London Press, 1962), p. 29.

[18]Getzels, J.W., "Creative Thinking, Problem-solving and Instruction," Theories of Learning and Instruction, Sixty-third Yearbook of the National Society for the Study of Education (Chicago: NSSE , 1964), pp. 240-268.

[19]Evans, op. cit., p. 20.

[20]Brune, I.H., "Geometry in the Grades," The Arithmetic Teacher, VIII (May, 1961), p. 211.

[21]Hohn, F.E., "Teaching Creativity in Mathematics," The Arithmetic Teacher, VIII (March, 1961), p. 103.

[22]Kemeny, J.G., "Rigor vs. Intuition in Mathematics," The Mathematics Teacher, LIV (February, 1961), p. 71.

and Lankford[23] claim that at least one of the common goals of all teachers of mathematics should be to develop in students their abilities to create new ideas. Hildebrandt stated that the real test of an individual's worth is not merely his mathematical knowledge, but how well he can suggest or discover methods of solution when confronted with a problem situation. He says that "all students should repeatedly and continuously be held to discover or invent mathematical concepts and ideas for themselves."[24] Polya too recognized this aspect of mathematics teaching:

> The teaching of mathematics should acquaint the students with all aspects of mathematical activity as far as possible. Especially, it should give opportunity to the students for independent creative work.[25]

Evans suggests that teachers have an important part to play in the development of the creative mathematical abilities of their students. "Thus it appears that, if they are provided with experiences appropriate to their mathematical preparation, students at all grades are able to discover

---

[23]Lankford, F.G. JR., "Implications of the Psychology of Learning for the Teaching of Mathematics," The Growth of Mathematical Ideas, K-12, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1959), p. 405.

[24]Hildebrandt, E.H.C., "Mathematical Modes of Thought," The Growth of Mathematical Ideas, K-12, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1959), p. 371.

[25]Polya, G., Mathematical Discovery (New York: Wiley, 1965), II, p. 143.

mathematical ideas."[26] He further states that intuitive thinking ability is an essential ingredient of mathematical creativity.[27] He is supported in this view by Nidditch who thinks that almost all mathematicians would agree that intuition, whatever it is, may well be the source of new mathematics.[28] Popper quotes Einstein's response to the question of whether he felt there were highly universal laws from which a picture of the world can be obtained by pure deduction:

> There is no logical path leading to these . . . laws. They can only be reached by intuition.[29]

Polya too thinks that a student needs to learn to use intuitive thinking since it is on this kind of thinking that his creative work will depend.[30]

Thus we see that the phrase "intuitive thinking" is used extensively in connection with mathematical creativity. In this context, intuitive thinking requires the ability to produce combinations of ideas already possessed by the learner--not just any combination, but those which have a

---

[26] Evans, op. cit., pp. 199, 201.

[27] Ibid., p. 40.

[28] Nidditch, P.H., Elementary Logic of Science and Mathematics (Glencoe, Ill.: The Free Press of Glencoe, 1960), pp. 302-303.

[29] Popper, K.R., The Logic of Scientific Discovery (New York: Basic Books, Inc., 1959), pp. 31-32.

[30] Polya, Mathematics and Plausible Reasoning, I, p. vi.

mathematically useful potential and are, therefore, satisfying. This would suggest extensive familiarity with the basic ideas and with the structure of mathematics so that those combinations which are meaningful and interesting can be recognized.

Learning Efficiency. Every teacher strives to use instructional procedures which result in learning. Moreover, we are concerned about achieving this goal with the greatest efficiency possible--not only in the immediate future, but for a life-time. We are not only concerned with the time element, but also with the quality of learning. Rote-learning is certainly not efficient nor sufficient. Only complete understanding of mathematical concepts is acceptable. Many writers believe that intuitive thinking can greatly increase learning efficiency.

After describing Inhelder's "pre-curriculum" devoted to developing intuitive understanding in children for the first two or three years of formal schooling, Bruner quotes her as saying:

> The effect of such an approach would . . . be to put more continuity into science and mathematics and also to give the child a much better and firmer comprehension of the concepts which, unless he has this early foundation, he will mouth later without being able to use them in any effective way.[31]

Bruner goes on to state that many of the most highly regarded teachers in mathematics and science strive to develop

---

[31]Bruner, The Process of Education, p. 46.

effectiveness in intuitive thinking. Curriculum improvements direct at least some of their efforts to the development of procedures that will contribute to the improvement of intuitive thinking.[32]

Willoughby points out that top-notch mathematicians find formalism without a good intuitive understanding of the basic concepts underlying the formalism inadequate.

> In a similar way, no school pupil can be expected to understand and appreciate good mathematics unless adequate informal background is laid for formally expressed thoughts . . . Indeed, the teacher should never allow formalism to take the place of well-developed intuitive understandings.[33]

Ausubel[34] and Lovell[35] state that intuitive thinking is particularly appropriate, and even necessary, in the early stages of the development of a concept. First impressions are lasting, and if these are merely mouthed phrases about a concept with no mental picture of it, the student is often satisfied that he "knows" the concept. Dienes too attributes successful learning to the development of a "construtive" (intuitive)process. This will be described more fully in Chapter III.

Intuitive thinking as used in the context of learning

---

[32]Ibid., pp. 56, 57.

[33]Willoughby, S.S., Contemporary Teaching of Secondary School Mathematics (New York: John Wiley and Son, Inc., 1957), p. 225.

[34]Ausubel, op.cit., p. 119.

[35]Lovell, op. cit., p. 29.

efficiency implies a thought process which is able to give mental images of concepts quickly and effectively. The process by which the concept is attained is not important at this stage, but rather that a mental image is created of the concept.

Elements of Intuitive Thinking. One of the problems in coming to grips with the concept of intuitive thinking is defining it. The problem is even more acute if one sets out to define it in behavioral terms, that is, in terms of actual observable events. In this study, these observable events which are to be taken as manifestations of intuitive thinking are called "elements of intuitive thinking," a phrase borrowed from Evans.[36]

Evans asks the following questions:

> Are there certain kinds of mental activity that can be classified as intuitive? Can [intuitive thinking] be described in terms of subprocesses? That is, are there certain observable techniques used by the intuitive thinker which, if identified, would be of aid in studying the process?[37]

It is the purpose of this section to identify observable methods and techniques used by the intuitive thinker as he functions in a classroom situation. The intuiter may be either the teacher or the student. Though there may be significant unobservable elements of intuitive thinking, these will not be of any assistance to us since we are taking the

---

[36]Evans, op. cit. p. 27.

[37]Ibid.

direct approach in this study rather than the products-of-the-process approach.

Bruner,[38,39] Evans,[40] Polya,[41,42] Dienes,[43,44] Burton,[45] and Guilford[46] have listed one or more of the following as elements of intuitive thinking (at this point these will be identified and in a later section an attempt will be made to reduce these to simplest possible terms so that there will be minimum overlapping among the elements):

1. Symmetry of a structure - the number of automorphisms, that is self mappings, in the structure.

---

[38]Bruner, The Process of Education, pp. 60-65.

[39]Bruner, Jerome S., On Knowing (Cambridge: Belknap Press of Harvard University Press, 1962), pp. 103-104.

[40]Evans, op. cit., pp. 10, 14-15, 26-29.

[41]Polya, Mathematics and Plausible Reasoning, I, pp. vi-viii, iii, 3-5, 7, 22, 26-28, 35, 50, 55, 12-13, 111, 190, 204, II, p. 158.

[42]Polya, How to Solve It, p. 63.

[43]International Study Group to Mathematics Learning, Mathematics In Primary Education (Hamburg: Unesco Institute for Education, 1966), pp. 54-55.

[44]Dienes, Z.P., Building up Mathematics (London: Hutchinson Educational, 1960), p. 35.

[45]Burton, W.H., R.B. Kimball and R.L. Wing, Education for Effective Thinking (New York: Appleton-Century-Crofts, Inc., 1960), p. 407.

[46]Guilford, J.P., P.R. Merrifield, and A.B. Cox, Creative Thinking in Children at the Junior High School Levels, Cooperative Research Project No. 737, Reports from the Psychological Laboratory, No. 26 (Los Angeles: University of Southern California, September, 1961), p. 5.

2. Experimentation - the examination and testing of possible solutions.

3. Intelligent trial and error - attempting things with a purpose in mind.

4. Guessing - purposeful guessing, with testing of guesses.

5. Successive approximations.

6. Manipulation - engaging in activity with elements of the problem.

7. Concrete materials - embody mathematical structures in concrete materials, stories, or games.

8. Examining limiting conditions of a problem (not the solution).

9. Redefinition - rearranging the given information into a form which may suggest a solution.

10. Similarity - recalling related problem situations and making use of the techniques.

11. Analogy - comparing the problem situation to a concrete embodiment of the concept.

12. Visualizing the solution.

13. Visualizing the finished structure.

14. Generalization - extending techniques from given situation to a more general case.

15. Specialization - examining a special case to gain ideas for proofs and solutions of general case.

16. Using concepts or principles without explicit verbalization of the same.

17. Skipping steps in arriving at a solution.

18. Substituting a visual proof for a formal proof.

19. Induction - the examination of specific facts and the seeking of possible generalizations based on these facts.

20. Comparison - comparing the elements of the problem situation with respect to such criteria as shape, size or position.

21. Emphasis on the structure of knowledge - familiarity with basic concepts.

IV. COMPLEMENTARY NATURE OF INTUITIVE AND ANALYTICAL THINKING

Earlier reference was made to a useful function of analytic thinking. Though it is useful to use the words analytic thinking and intuitive thinking separately, one must not fall into the error of holding forth the merits of one process over against the merits of the other.

Evans,[47] Polya,[48] and other writers have recognized the complementary nature of intuitive and analytic thinking. Bruner, for example says:

> The complementary nature of intuitive and analytic thinking should, we think, be recognized. Through intuitive thinking the individual may often arrive at solutions to problems which he would not achieve at all, or at least slowly, through analytic thinking. Once achieved by intuitive methods, they should if possible be checked by analytic methods.[49]

---

[47]Evans, op. cit., p. 23.

[48]Polya, How to Solve It, p. vii.

[49]Bruner, The Process of Education, p. 59.

Nidditch states that the products of intuition are only a beginning and not the end. They have to be checked by a deductive process.[50] In speaking of scientific discovery, Popper distinguishes between the process of conceiving a new idea ("creative intuition") and the methods and results of examining it logically (analytic thinking).[51] Hadamard too supports the idea that intuitive and analytic thinking are complementary: "Some intervention of intuition issuing from the unconscious is necessary at least to initiate the logical work."[52]

We note that the complementary nature of intuitive and analytic thinking is one of order, intuition preceding analytic thinking. Thus it would seem that when learning a new concept one could speak of an intuitive phase and an analytic phase. Furthermore, for purposes of research it would be profitable to concentrate exclusively on one phase. This study is concerned with the intuitive phase.

V. DEVELOPMENTAL ASPECTS OF INTUITIVE AND ANALYTICAL THINKING

There are at least two vantage points from which to

---

[50]Nidditch, op. cit., p. 303.

[51]Popper, op. cit., p. 31.

[52]Hadamard, Jacque, An Essay on the Psychology of Invention in the Mathematical Field (Princeton, N.J.: The Princeton University Press, 1949), p. 112.

view developmental aspects of intuitive and analytical thinking. One can look at the chronological development of the mental capacities and, at the development of individual concepts.

Chronological Development of Mental Capacities. The sequential aspects of Piaget's three stages of mental development--the preoperational stage, the concrete operations stage and the formal operations stage--have been supported by research. Dienes,[53] Hull,[54] and Biggs[55] have presented supporting evidence.

Intuitive thought processes are already functioning during the preoperational stage, but it first becomes the prevailing mode of thought during the concrete operations stage. In this stage children are able to engage in analytic thinking when problems are presented in concrete situations. During the formal operations stage the child is capable of analytic thinking in situations not closely related to reality. Intuition is active in the formulation of hypotheses which he is now able to test analytically.[56] In reviewing the literature on this subject, Evans states

---

[53] International Study Group for Mathematics Learning, op. cit., p. 57.

[54] Ibid.

[55] Biggs, J.B., "The Development of Number Concepts in Young Children," Educational Research, I (February, 1959), pp. 17-34.

[56] Evans, op.cit., pp. 31,37.

that:

> The evidence indicates that a child can learn, by an intuitive process, many mathematical concepts long before he can use these concepts in processes of logical deduction, that is, analytical thinking. Since the child is able to learn intuitively in his early school years, it seems reasonable to expect that a great deal of attention should be given to this process, and that effort should be made to assist the child in the development of his intuitive ability.[57]

Bruner too warns against attempting formal explanations based on a logic that is distant from the child's manner of thinking. "Poor teaching results from premature formalism."[58]

Ausubel too points out that the elementary school child "is largely restricted to a subverbal, concrete, or intuitive level of cognitive functioning." However, he goes on, the Junior High School student can transcend the previously achieved level of subverbal, intuitive thought and understanding. He is now capable of greater clarity, precision, explicitness and generality associated with advanced stages of intellectual development.[59]

Concept Development. Lovell states that in the teaching of mathematics, intuition brings the first ideas to the child.[60] In this sense intuition is thought of as a phase in the development of a concept. Earlier, reference was already made to this aspect of concept development.

---

[57] Ibid., p. 35.

[58] Bruner, The Process of Education, p. 38.

[59] Ausubel, op. cit., pp. 117-118

[60] Lovell, op. cit., p. 29.

According to Ausubel, each individual undergoes a "transition from concrete to abstract functioning in _each_ new subject area, even after he reaches the abstract stage." When a learner first encounters a wholly unfamiliar subject-matter field, he initially tends to function at a concrete, intuitive level. He is able to pass through this stage more rapidly than young children. Ausubel emphasizes, though, that intuitive thinking should be used only during the _early_ stages of concept development.[61]

Again, Dienes has a similar view when he speaks of his "cycles of learning." In Chapter III reference will be made to the intuitive and analytic phases of concept development in the context of Dienes' "learning cycles."

VI. A BEHAVIORAL DEFINITION OF INTUITIVE THINKING

The approach taken here in presenting a behavioral definition of intuitive thinking is similar to that taken by the people who define intelligence to be what the test measures. The categories of the instrument form the basis of the definition presented in this section. Thus the first task undertaken is an examination of the twenty-one previously listed elements of intuitive thinking with the purpose of establishing distinct, clearly observable, and mutually exclusive elements of intuitive thinking. These are at the same time the categories of the instrument and, therefore,

---

[61]Ausubel, _op. cit._, pp. 119, 132.

are the behavioral items in the definition of intuitive thinking.

In reading various authors who have written about one or more of the elements of intuitive thinking, the investigator very often found a confusing array of definitions referring to the same term. In trying to sort out distinct elements of intuitive thinking, the investigator found it most advantageous to group the definitions which seemed to overlap or define the same thing and then attach a term to the definition. The terms may have different meanings from those previously cited. However, for the benefit of the reader of this study, it seemed more convenient to name the elements of intuitive thinking (that is, the name ultimately selected) first and then give the defining characteristics of the element, with supportive statements from various authors.

1. Symmetry. Dienes defined symmetry to be "the number of automorphisms, that is self-mappings, in the structure."[62] Polya says that "a whole is symmetric if it has interchangeable parts."[63] He goes on to say that if a problem or structure is symmetric in some ways it may be profitable to notice its interchangeable parts and to treat those parts which play the same role in the same fashion. Webster defines

---

[62]International Study Group for Mathematics Learning, op. cit., p. 54.

[63]Polya, How to Solve It, p. 19.

symmetry to be the correspondence in size, shape, and relative position of parts that are on opposite sides of a dividing line or median plane or that are distributed about a center or axis. Without defining the term, Bruner claims that the appeal to symmetry will be a "support to intuitive thinking."[64]

2. Analogy. Again without defining the term, Bruner says that the use of analogy will be "a support to intuitive thinking."[65] The investigator found confusion in the use of the term analogy, comparison, and similarity and arbitrarily separated certain characteristics which he felt were sufficiently distinct and formed two elements: analogy and similarity. Polya says that:

> Analogy is a sort of similarity. It is, we could say, similarity on a more definite and more conceptual level. . . The essential difference between analogy and other kinds of similarity lies, it seems to me, in the intentions of the thinker. Similar objects agree with each other in some aspect. If you intend to reduce the aspects in which they agree to definite concepts you regard those similar objects as analogous . . . Two systems are analogous, if they agree in clearly definable relations in their respective parts.[66]

In other words, he considers analogy and similarity variations of the same thing, but differing in degree. However, in this study, analogy refers to a procedure in which a comparison between a problem situation or a mathematical

---

[64]Bruner, The Process of Education, p. 64.

[65]Ibid.

[66]Polya, Mathematics and Plausible Reasoning, I, p. 13.

concept, and a different embodiment is made. The embodiment may be a concrete situation, a Dienes' story or game, or another mathematical structure. It includes Polya's idea of analogy and similarity, as well as the idea of comparison. The inference is made that if two things agree in one or more aspects, they will probably agree in yet other aspects. On the idea of concrete embodiments as an element of intuitive thinking, Bruner says that:

> Perhaps the first thing that can be said about intuition when applied to mathematics is that it involves the embodiment or concretization of an idea, not yet stated, in the form of some sort of operation or example.[67]

The term similarity was reserved for another definition.

3. Similarity. The difficulties concerning the use of the terms similarity and analogy have already been referred to above. The term similarity was reserved for the procedure of recalling related problem situations having characteristics in common and making use of the techniques used there. This differs from analogy in that in analogy we are making a comparison between a mathematical concept or structure and the embodiment of a concept or structure, whereas in similarity we are making a comparison between different problem situations having common elements. We could be making a comparison between two problem situations at the

---

[67]Bruner, On Knowing, p. 103.

same level of abstraction--for example, both could be different embodiments of the same concept--and call it a similarity. Again, as in analogy, the inference is made that if two things agree in one or more respects, they will probably agree in yet other aspects. Of similarity Polya says that "it may be easy to imitate the solution of a problem when solving a closely similar one."[68] By posing the following question, Bruner suggests that it is important to use similarity to encourage intuitive thinking:

> Should students be taught explicitly, 'When you cannot see how to proceed with the problem, try to think of a simpler problem that is similar to it; then use the method for solving the simpler problem as a plan for solving the more complicated problem'?[69]

4. Generalization. The term generalization was chosen to include the ideas conveyed by writers using the terms generalization, specialization and induction. Polya says that induction, distinct from mathematical induction "is the process of discovering general laws by the observation and combination of particular instances."[70] He goes on to say that induction finds regularity and coherence behind the observations. Generalization, says Polya, "is passing from the consideration of a given set of objects to

---

[68]Polya, G., Mathematical Discovery (New York: John Wiley and Sons, 1962), I, p. v.

[69]Bruner, The Process of Education, p. 63.

[70]Polya, How to Solve It, p. 114.

that of a larger set, containing the given one."[71] By specialization he means "passing from the consideration of a given set of objects to that of a smaller set, contained in the given one."[72] Evans has defined the terms as follows:

> Specialization: examining cases (such as numerical examples) to receive ideas for proofs or solutions. Generalization: extending the techniques used in a given situation to more general situations. Induction: the examination of specific facts and the seeking of generalizations based on these facts.[73]

He also says that these are elements of intuitive thinking. Bruner notes that induction and intuition are used by mathematicians and suggests that there may be a relationship between the two.[74] In this study we are defining intuitive thinking to include the inductive process.

5. Experimentation. The term experimentation was chosen to include the ideas of experimentation, intelligent trial and error, purposeful guessing, and successive approximations. Evans defined these terms as follows:

> Experimentation: the examination and testing of possible solutions in light of the given information. Guessing or playing hunches: while this type of activity is not to be encouraged as the only technique, checking on one's guesses may be of assistance in developing ideas. Intelligent trial and error: attempting things with a purpose in mind. This might also be considered as the method of successive approximations.[75]

---

[71]Polya, Mathematics and Plausible Reasoning, I, p. 12.

[72]Ibid., p. 13.

[73]Evans, op. cit., pp. 27-28.

[74]Bruner, The Process of Education, p. 66.

[75]Evans, loc. cit.

Again Evans includes these as elements of intuitive thinking. Polya uses these terms interchangeably. Trial and error, he says,

> consists of a series of trials, each of which attempts to correct the error committed by the preceding and, on the whole, the errors are diminished as we proceed and the successive trials come closer to the desired final result. . . We may speak of successive approximations . . . The teacher should not discourage his students from using trial and error.[76]

On the subject of guessing Bruner states that:

> The teacher who is willing to guess at answers to questions asked by the class and then subject his guesses to critical analysis may be more apt to build intuitive methods into his students than would a teacher who analyzes everything in advance for his class.[77]

6. Manipulation. By manipulation as an element of intuitive thinking Evans means the "engaging in activity with the elements of the problem. Activity, even if not meaningful at the start, may help in initiating the flow of ideas."[78] In Chapter III, it will be pointed out that Dienes' concept of constructive thinking and Bruner's concept of intuitive thinking are essentially the same. Thinking of constructive thinking as being intuitive thinking, Dienes places great emphasis on "fiddling about" with the elements of a problem. This aspect of manipulation is documented more fully in

---

[76]Polya, Mathematical Discovery, I, p. 26.

[77]Bruner, The Process of Education, p. 62.

[78]Evans, loc. cit.

Chapter III. Also included under the term manipulation is Bruner's procedure of "examining the limiting conditions of a problem" (but not of the solution), which he also lists as one of the things which "support intuitive thinking."[79]

7. Skipping steps. By skipping steps is meant the solving of a problem in which many of the logical steps of the argument are apparently omitted or subconsciously used. This element was derived directly from a consideration of the various definitions of intuitive thinking presented earlier. One of the defining characteristics given was the skipping of steps while working out a solution.

8. Pre-verbalization. By pre-verbalization is meant the use of concepts or principles without the explicit verbalization of the same. Bruner states that he is

> objecting to something far worse, the premature use of the language of mathematics, its end-product formalism . . . At our worst, we offer formal proof (which is necessary for checking) in place of direct intuition.[80]

Gertrude Hendrix put much stress on the distinction between verbalizing and having developed a mathematical concept. She contended that a student can discover mathematical results without being able to verbalize them.[81]

---

[79]Bruner, The Process of Education, p. 56.

[80]Bruner, On Knowing, p. 104.

[81]Hendrix, Gertrude, "A New Clue to Transfer of Training," Elementary School Journal, XLVIII (December, 1947), pp. 197-208.

9. Solution visualization. By solution visualization is meant a pre-occupation with the solution or finished structure in which attempts are made to delimit the possible magnitude, shape, position, or any other characteristics of the solution or finished product. No attempt is made at stating an actual solution. Bruner states that the "visualization of the solution" will be "a support to intuitive thinking."[82] Polya says that "the end suggests the means; the consideration of the aim may suggest an approach."[83]

10. Informal "proof." Informal "proof" refers to the procedure whereby a visual (or "inductive") proof is substituted for a formal proof. The classroom activity referred to here was considered to be sufficiently distinct from that described by the element "generalization" above since we are not concerned with generalizations here, but an actual "proof" satisfying to the student. There is no attempt to glean a pattern or manner of proof to be used in a formal proof. Bruner states that

> there has been very little done, for example, on the use of diagrams as geometrical experiments . . . in which visual proof substitutes for formal proof where possible.

He goes on to suggest that such activity would probably lead to a "better intuitive understanding."[84]

---

[82]Bruner, The Process of Education, p. 64.

[83]Polya, Mathematical Discovery, II, p. 79.

[84]Bruner, The Process of Education, p. 56.

We are now ready to give a behavioral definition of intuitive thinking as the term was used in this study. Intuitive thinking is thinking which makes use of the following processes (each of the terms being defined as above): symmetry, analogy, similarity, generalization, experimentation, manipulation, skipping steps, pre-verbalization, solution visualization, and informal proof.

CHAPTER III

A THEORY OF MATHEMATICS LEARNING AND INTUITIVE THINKING

I. INTRODUCTION

In Chapter I reference was made to Bruner's concern over the devaluation of the role of intuition in school learning. It is the purpose of this Chapter to place intuitive thinking in the context of a theory of mathematics learning developed by Z. P. Dienes and in so doing suggest a plausible function of intuitive thinking in the process of learning a mathematical concept. It is hoped that the successful attainment of this purpose will help to dispel some of the suspicion surrounding the use of an intuitive approach in instruction.

Before taking up the main subject of this chapter it will be necessary to present some background information on contemporary learning theories and on Dienes' conception of mathematics and mathematics learning.

II. MATHEMATICS AND MATHEMATICS LEARNING

By mathematics, Dienes means "the actual structural relationships between concepts connected with numbers (pure mathematics) together with their applications to problems arising in the real world."[1] Dienes goes on to point out

---

[1] Dienes, Z.P., Building Up Mathematics (London: Hutchinson Educational, 1960), p. 31.

that when we consider

> what we mean by an ability to engage in a mathematical activity, we must be clear about the existence of at least two aspects of such activity: (i) disentangling the conceptual structure and (ii) acquiring and using the techniques involved in operating and applying the structure.[2]

This then implies two types of abilities: the ability to understand and the ability to handle techniques.

By the learning of mathematics Dienes means the apprehension of the structural relationships together with their symbolization, and the acquisition of the ability to apply the resulting concepts to real situations occurring in the world.[3]

In order for a child to develop mathematical concepts he must know the language and symbols of mathematics. Lovell suggests that besides understanding concepts, children must learn to combine them with other concepts, using the language and symbols, that is, mathematics is a mental activity.[4] Dienes holds a similar view:

> Mathematics is based on experience; it is the crystallization of relationships into a beautiful regular structure, distilled from our actual contacts with the real world.[5]

---

[2]Dienes, Z.P., Concept Formation and Personality (London: Leicester University Press, 1965), p. 2.

[3]Dienes, Building Up Mathematics, pp. 31-32.

[4]Lovell, K., The Growth of Basic Mathematical and Scientific Concepts in Children (London: University of London Press, 1962), p. 20.

[5]Dienes, Building up Mathematics, p. 10.

It is questionable whether most of the mathematics taught in our schools is of the type described above. A utilitarian value can probably be attached to mathematics as described above, but it is not the purpose here to advocate utilitarian or academic aims of mathematics education. However the theory of mathematics learning to be presented hereafter will have some significant implications for teachers who want to teach mathematics with understanding, rather than being content with rote learning.

III. BACKGROUND INFORMATION ON CONTEMPORARY LEARNING THEORIES

Much of the learning theory presently in vogue has been based upon animal experimentation in which the teaching process has been neglected. On the whole these learning theories are too simple and elementary to account for the extremely complex processes that must go on in the mind of anyone trying to acquire complex structures of the sort which abound in mathematics. Contemporary learning theories are not adequate for explaining or suggesting experiments relating to complex mental processes. Buswell's comment on this issue is worth noting:

> Without the slightest criticism of experiments in general psychology, we cannot continue to be satisfied with implications for education from results of experiments with simple mental processes, with animals, and at the sub-language level . . . I am proposing that educational psychologists take their cues for research from the problems of learning in schools where the processes are complex and where the learnings carried

on are at the language level.[6]

Piaget is a researcher who has studied the learning process in its full complexity. He has spent many hours studying children in actual learning situations. Two of his ideas are particularly significant to Dienes' theory of mathematics learning and thus to our conception of the function of intuitive thinking in learning mathematics concepts. The first of these is "thought is internalized action." Piaget believes that all the child's learning has its basis in his own activity as he interacts with his physical and social environment.[7]

> In effect thought itself in this view is simply an internal version or development of outward action. It is action which becomes progressively internalized through the child's acquisition of language and his growing use of symbols, through imagination and representation.[8]

It then follows that the child must actively construct his own thought system.

> According to Piaget, perception is not the mere passive registering of raw sensations by the child's sense organs. It is to be understood as perceptual activity, in which the child's brain organizes the

---

[6]Buswell, G.T., "Educational Theory and the Psychology of Learning," The Journal of Educational Psychology, XLVIII (March, 1965), p. 181.

[7]Adler, Irving, "Mental Growth and the Art of Teaching," The Mathematics Teacher, LIX (December, 1966), p. 707.

[8]Isaacs, Nathan, New Light on Children's Ideas of Numbers: The Work of Professor Piaget (London: Educational Supply Association, 1960), p. 6.

> sensations he gathers in the course of his exploratory activity.[9]

The second idea is his concept of assimilation and accomodation. Irving Adler has summarized the concepts as follows:

> Mental activity, like metabolic activity, is a process of adaptation to the environment. Adaptation consists of two opposed but inseparable processes, assimilation and accomodation. Assimilation is the process whereby the child fits every new experience into his pre-existing mental structures . . . Accomodation is the process of perpetual modification of mental structures to meet the requirements of each particular experience.[10]

The structure of a body of knowledge present in the mind at any given time Piaget calls a schema. Learning then becomes a matter of incorporating perceptions into the schema or enlarging and modifying the existing schema.[11]

Skemp advocates a similar approach to learning. In an article entitled "The Need for a Schematic Learning Theory" he makes the case that an adequate learning theory must take into account (among other things) the systematic development of an organized body of knowledge.[12] In all subjects, new learning is made possible by some of the knowledge we already

---

[9]Adler, <u>op</u>. <u>cit</u>., p. 709.

[10]<u>Ibid</u>., p. 707.

[11]Skemp, R.R., "The Need for a Schematic Learning Theory," <u>The British Journal of Educational Psychology</u>, XXXII (February, 1965), p. 133.

[12]<u>Ibid</u>.

have, but this varies from subject to subject. Skemp points out that "in mathematics dependence on earlier knowledge is great."[13] This is not to say that non-schematic learning is impossible. Indeed Skemp laments the fact that students can rote-learn what is required of them. When this happens, not only is understanding absent, but the schema required for future learning is not being prepared.[14] Biggs, too, builds on the ideas of Piaget. He uses the ideas of structured and unstructured (alogical) learning to refer, respectively, to learning in which there is a balance between assimilation and accomodation and learning in which we have only assimilation (that is, rote learning).[15]

IV. DIENES' THEORY OF MATHEMATICS LEARNING

Z. P. Dienes has developed a theory of mathematics learning based on many of these ideas and on his own extensive experience with children learning mathematics concepts. Two events must take place when mathematics is learned:

> (i) sorting events into classes or categories, so that any event is immediately recognized as either

---

[13] Skemp, R.R., "A Three-part Theory of Learning Mathematics," New Approaches to Mathematics Teaching, F.W. Land, editor (London: MacMillan, 1963), p. 45.

[14] Ibid., p. 47.

[15] Biggs, John B., "Towards a Psychology of Educative Learning," International Review of Education, XI (January, 1965), pp. 77-79, 90.

> belonging or not belonging to a class or category, or of course, as being irrelevant to it;
> (ii) becoming aware of the relationship to each other of the classes or categories constructed.[16]

The first event constitutes forming an element to class relationship and the second constitutes forming a class to class relationship. This is clearly reminiscent of Piaget's statement "the child organizes the sensations he gathers in the course of his exploratory activity" referred to earlier in the chapter. The second event referred to by Dienes refers to the influence of prior learning on present concept development. The events also suggest the building up and adjusting of mental schema (that is, the assimilation and accommodation referred to earlier in the chapter).

Dienes has developed several principles of mathematics learning. The first of these he calls the dynamic principle.[17] Learning takes place in three stages to form a "cycle" which is repeated for each new concept learned. The product of earlier cycles form the objects of the new situation initiating another cycle of learning. The three stages are:

(i) Play stage. This is a time of random manipulating of the new situation. Dienes uses the phrase "fiddling about" to characterize this stage. In this way the learner is laying the foundation on which he will

---

[16]Dienes, Z.P., The Power of Mathematics (London: Hutchinson Educational, 1964), pp. 21-22.

[17]Ibid., pp. 23-32.

structure his own thought systems.

(ii) Learning the rules. From the play stage experiences, the restrictions of the new situation make themselves felt and we are led to the realization of the regularities inherent in the situation. Though we may not agree with the Gestaltist position that this search for regularity is innate or that, indeed, an objective regularity exists in nature, we cannot deny that the search for regularity by human beings exists. In this stage the learner is actively involved in building his own thought systems.

(iii) Reification. In this stage we treat an abstraction as substantially existing or as a concrete object. It is now that we say "we understand" or "we have gained insight." We consolidate this new concept in one or both of two ways: (a) practice (b) analysis. Practice leads to a more or less unconscious stamping in of the regularity so that we are able to use it as a matter of course. The structure "soaks into us as one more way of sorting out our environment."[18] Analysis may be one of two kinds: (1) retroactive analysis (2) progressive analysis. Retroactive analysis is looking back at the structure that has been built and analyzing it in a critical way. This is usually a conscious activity. Progressive

---

[18] Ibid., p. 30.

analysis is an extension of the rule structure, that is, generalizing. It is an "extension of an already existing class into a more extensive class."[19]

To increase the efficiency of mathematics learning (that is, the operation of the cycle), Dienes has formulated three more principles.[20] The first principle is the principle of multiple embodiment. To allow as much scope as possible for individual variations in concept formation, as well as to induce the learner to gather the mathematical essence of an abstraction, the same conceptual structure should be presented in as many perceptually equivalent forms as possible. Mental growth is encouraged by the experience of seeing things from many points of view. Although this is especially important for the young child, it should not be neglected in teaching the older child and the adolescent. This ensures that exemplars of the concept will be recognized. The principle of contrast requires that both exemplars and non-exemplars of a concept be presented. This will ensure that situations not possessing the structure will be recognized as such. The mathematical variability principle states that the variables involved in the situation should be varied. This will focus attention on the essentially invariant features.

Since our natural environment provides few opportunities for children to manipulate mathematical concepts and

---

[19] Ibid., p. 32. [20] Ibid., pp. 40-41.

> that takes place without the thinker being aware of the details of what he is thinking about. He may arrive at a correct conclusion by looking at the whole structure. He may then be realizing in some sense how the structure works without being aware of the internal relationship of the structure.[25]

This concept of constructive thinking has the same elements as Bruner's concept of intuitive thinking described in Chapter II. This then suggests how the function of intuitive thinking can be described in the context of Dienes' theory of mathematics learning.

Dienes emphasizes the importance of giving children a wide range of experience with mathematical ideas, even to the extent of random manipulation of the elements of a mathematical concept without any specific goal in the mind of the child. The purpose of this is to give the learner a wide range of meaningful imagery which he can employ in completing the learning cycle. The emphasis here is to give the learner an intuitive feeling of the situation without having to describe or consider all the relationships among the elements he is manipulating. Dienes feels that if the only mathematical structures available to the learner are those which he builds deductively, he is sure to be mentally impoverished. When a child learns a language, he has a rich environmental background to provide him with mental images,

---

[25]Dienes, Z.P., "Research in Progress," New Approaches to Mathematics Teaching, F.W. Land, editor (London: MacMillan, 1963), p. 51.

but when he begins to learn mathematical structures he does not have such an abundant supply of referents.[26] It is the belief of this investigator that constructive thinking or intuitive thinking can provide an abundant supply of meaningful mental images of mathematical structures.

Furthermore, Dienes and Piaget contend that before the age of eleven or twelve, the child is practically incapable of analytic thought and must rely heavily on intuitive thinking.[27] Even later, when the learner is first introduced to a wholly unfamiliar concept, he initially tends to function at a concrete, intuitive level.[28] Lovell maintains that "at any stage in the teaching of mathematics, experience and intuition bring the first ideas to the child."[29] Willoughby recognizes that formalism has a place in mathematics education, but maintains it should "not be allowed to usurp the place of understanding and intuition."[30]

In summary then, the poverty of mathematical imagery in most children's minds is an important factor in their

---

[26] Ibid.

[27] Dienes, Z.P., "The Growth of Mathematical Concepts in Children Through Experience," Educational Research, II (November, 1959), p. 12.

[28] Ausubel, D.P., The Psychology of Meaningful Verbal Learning (New York: Grune and Stratton, 1963), p. 119.

[29] Lovell, op. cit., p. 29.

[30] Willoughby, S.S., Contemporary Teaching of Secondary School Mathematics (New York: Wiley, 1967), p. 225.

difficulties in coping with mathematics. Intuitive thinking, therefore, can be and should be used to increase the efficiency of mathematics learning and to increase mathematical creativity. Bruner gives his support to the importance of this conviction in the following statement:

> It is important to allow the child to use his natural and intuitive ways of thinking, indeed to encourage him to do so, and to honor him when he does. I cannot believe that he has to be taught this. Instead, we should first end our habit of inhibiting intuitive thinking and then find ways of helping the child improve at it.[31]

---

[31]Bruner, <u>On Knowing</u>, p. 105.

CHAPTER IV

THE REVIEW OF SELECTED RESEARCH

I. INTRODUCTION

The writer was unable to find any research which attempted to develop an instrument for measuring intuitive thinking in the classroom. However, in recent years considerable effort has been expended in trying to measure classroom behavior by systematic observation. Though this is an obvious approach to research in teaching, until recently it has been attempted only rarely. Typically, antecedents or consequences or both have been studied and inferences made as to the nature and quality of the teaching used. Oftentimes, it was merely assumed that a teacher would teach by certain methods when instructed to do so.

Direct research into intuitive thinking is rare indeed. There have been some attempts at obtaining empirical evidence on the process, though the focus has been on the thought processes of the student and not on the interaction between pupils and teacher.

Accordingly, in this chapter a review of selected research on the methodology of instrument development for measuring classroom behavior and on empirical studies attempted with intuitive thinking will be presented.

II. MEASUREMENT OF CLASSROOM BEHAVIOR BY SYSTEMATIC OBSERVATION

The first need for measuring classroom behavior arose in connection with the function of supervisory personnel in public school systems.[1] Since then the technique was found to be useful for measuring effective teacher behavior, measuring classroom climate, measuring multiple dimensions of classroom behavior and in classroom experimentation.[2]

One problem with many of these instruments has been their failure to get at any aspect of classroom behavior related to pupil achievement of cognitive objectives. Medley and Mitzel state that two studies have taken important first steps toward the construction of such instruments. These are the studies done by B.O. Smith and by Muriel Wright and Virginia Proctor.[3] The emphasis in the review of these studies will be on the methodology of the development of the instruments.

The B.O. Smith Study.[4] This study had its beginnings

---

[1]Medley, D.M., and H.E. Mitzel, "Measuring Classroom Behavior by Systematic Observation," *Handbook of Research On Teaching*, N.L. Gage, editor (Chicago, Rand McNally, 1963), p. 254.

[2]*Ibid*., pp. 237-290.

[3]*Ibid*., pp. 286-290.

[4]Meux, M. and B.O. Smith, "Logical Dimensions of Teaching Behavior," *Contemporary Research on Teacher Effect-*

with experiments which attempted to determine the effects of instruction in the logic of high school subjects on the ability of students to think critically. With the instrument that Smith is developing, he hopes to be able to answer the question of how one can describe the kind and amount of instruction given in the logic of subjects.

One of the stated purposes of Smith's study which is relevant to this study is "to devise a procedure for finding out whether or not there are logical dimensions of teaching and for describing such dimensions as may be found."[5] He is interested in observing, describing and classifying the phenomenon, with no effort being made to induce changes, to study its underlying conditions or to find its correlates. The appropriate modes of inquiry for this kind of research are those which attempt to distinguish among characteristics and those involved in the formulation of criteria defining the categories into which instances of the phenomenon are to be classified.

The selection of concepts to be used depends upon the standpoint from which teaching behavior is to be observed and classified. In Smith's study the elements to be studied were verbal behavior and the logical nature of behavior. To

---

iveness, B.J. Biddle and W.J. Ellena, editors (New York: Holt, Rinehart, and Winston, 1964), pp. 127-164.

[5] Ibid., p. 128.

analyze the mass and variety of verbal behavior of a classroom, two units were developed. The units to be chosen had to meet four criteria: low inference level, analyzable in terms of the logical aspects of teaching and of thinking, neutral with respect to subject matter and have a fair amount of reliability. The units of discourse chosen were the episode and the monologue. An episode is the one or more exchanges that comprise a completed verbal transaction between two or more speakers. The monologue is the solo performance of a speaker addressing a group.

Having selected the units into which the verbal discourse was to be broken down, Smith then established the criteria by which the individual units of discourse were to be identified. These are generally quite complex and are not presented here. (Smith says that they would be useless to any one who has not first developed a perceptual and technical background.) The criteria are divided into four groups. The group one criteria are concerned with identifying the opening phase of an episode (or the means of "entry" into episodes). Group two criteria and group three criteria cover the kind of remarks found in the sustaining and terminating phases of the episodes. The group four criteria are used to identify monologues.

The procedure for identifying the units of discourse using the above criteria involves a careful analysis of transcripts of the verbal discourse. In marking off the

transcripts, the analyst tags each utterance with one or more numbers taken from the set of criteria, each number representing the identity of the particular remark made.

The next step involves classifying the units (episodes and monologues) identified in the previous step. Some of the ways in which units may be classified are:

1. By the nature of their content.
2. The number of verbal interactions they involve.
3. By the psychological processes they involve.
4. By their logical features (the one Smith chose).

To classify the units requires establishing (1) categories and (2) criteria for deciding whether or not a given unit belongs to a particular category. Smith finally chose thirteen categories, some of which were broken up into subcategories: defining, describing, designating, stating, reporting, substituting, evaluating, opining, classifying, comparing and contrasting, conditional inferring, explaining, and directing and managing classroom. The criteria were first developed by trial and error. Each unit of a verbal discourse was written on a separate piece of paper and two investigators independently sorted the units into the thirteen categories, developing rules as they worked. The investigators then exchanged their rules and resorted the units on the basis of the rules before them. They then compared their classifications, noted the agreements and disagreements and made changes which evolved from their

deliberations. These rules were then used as the final criteria.

Since the present study is concerned with the development of an instrument, further procedures used by Smith to develop logical dimensions of verbal discourse are not reviewed.

The Wright-Proctor Study.[6,7] Whereas Smith limited his study to the logical processes of language, Wright studied three aspects of language and made these the basis for defining the categories of her instrument. Furthermore, Wright felt that the aspects of language studied by Smith, for example, would be put in proper perspective by developing categories and definitions from the species of mathematical language in the classroom. The three aspects of language considered by Wright were subject matter content, psychological process (the cognitive aspects) and sociological attitude (the affective aspect). Thus each unit of verbal discourse was classified with respect to content, process and attitude, that is, each unit of verbal discourse was considered to have

---

[6] Wright, Muriel J., "Development of an Instrument for Studying Verbal Behaviors in a Secondary School Mathematics Classroom," Journal of Experimental Education, XXVIII (1959), pp. 103-121.

[7] Wright, Muriel J. and Virginia Proctor, "Systematic Observation of Verbal Interaction as a Method of Comparing Mathematics Lessons," U.S. Office of Education Cooperative Research Project No. 816 (St. Louis, Mo.: Washington University, 1961).

three dimensions--content, process and attitude.

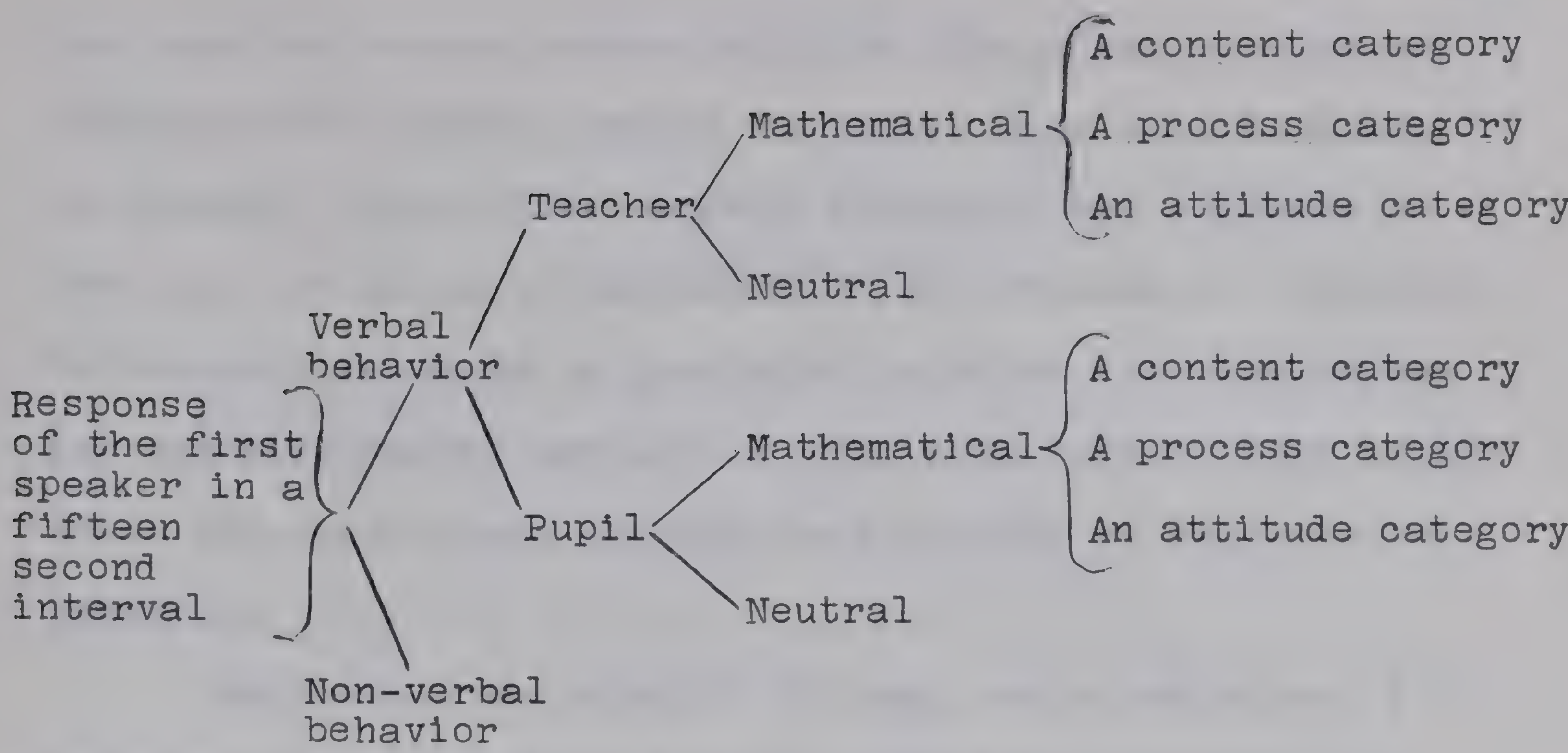
The categories of the content frame were selected to correspond to aspects of mathematical systems and functional classroom behavior. The question of "What aspects of mathematics is being worked on?" is answered by selecting a content category. The content categories were structure, technique, deductive, inductive, statement, mathematical application and other applications.

The categories of the process frame were developed on the basis of logical functioning in problem-solving. The syllogistic categories of analyzing and synthesizing include logical operations of inference. The classificatory categories of generalizing and specializing include the verbalization of generalizations, their applications,and the heuristic process of problem dissection and focussing on goals. The relevant category is a more static one. A statement of relevant information occurs when mathematical information is presented with no apparent logical sequence.

The categories of the attitude frame answer the question "How much initiative are the pupils asked to show, and how much do they demonstrate?" The categories of this frame are curiosity, independence, and receptivity.

A neutral dimension is provided to take care of non-verbal and non-mathematical units of verbal discourse.

Wright gives the following scheme for observation procedures:



III. MEASUREMENT OF INTUITIVE THINKING

The research to be reviewed in this section may seem to have little relevance to this study. It is being included to illustrate that there have been serious attempts to measure intuitive thinking empirically.

Guilford in a paper delivered to the Utah Conference on Creativity, stated that "much has been done in the direction of investigating operations under the heading of intuition or insight."[8] But he goes on to point out that many of these investigations were made through the examination of how conditions affect those phenomena.

Westcott presented a report to an earlier session of

---

[8]Guilford, J.P. "Basic Problems in Teaching for Creativity," Instructional Media and Creativity, C.W. Taylor, and F.E. Williams, editors (New York: Wiley, 1966), p. 89.

the same conference series entitled "Empirical Studies of Intuition."[9] In this report he states that some studies on insight, unconscious concept formation and transfer of training are actually addressed to the problem of intuition. He reports one study by Bouthilet in which learning curves for learning paired associates were constructed. The points where the curve rises sharply were thought to demonstrate intuition.

Westcott's own study[10,11] dealt with individual differences in the tendency to make intuitive judgments in solving problems.

Westcott describes his experimental situation:

> A subject is presented with a series of twenty problems to solve, and the information necessary to solve each is available in small quantities. The subject is instructed to solve as many problems as possible using as little information as possible. The actual format of the problems . . . consists of a masonite board with twenty rows of slots in it. Beneath each slot is a piece of information which I call a "clue," and each clue is covered by tin foil, which can be punched out with a stylus. The clues must be taken in the sequence given. When the person reaches what he feels is the answer he records it on the answer sheet.[12]

---

[9]Westcott, M.R., "Empirical Studies of Intuition," Widening Horizons of Creativity, C.W. Taylor, editor (New York: Wiley, 1964), pp. 34-53.

[10]Ibid.

[11]Westcott, M.R. "On the Measurement of Intuitive Leaps," Psychological Reports, XXI (1961), pp. 267-274.

[12]Westcott, "Empirical Studies of Intuition." p. 41.

He claims that he was able to identify intuitive thinkers with this instrument.

IV. CHAPTER SUMMARY

Research studies reviewed in this chapter provide support for the methodology for instrument development used in the present study and illustrate serious attempts at measuring intuitive thinking empirically.

The method used in constructing the instrument of the present study was patterned after the B. O. Smith study and is presented in detailed form in the following chapter. The work of Flanders[13] has not been reported since his instrument uses an alogical, timed unit. Such a unit is completely unsatisfactory for the study of thought processes. Bellack's "moves" and Kliebard's "teaching cycle" form an integral part of the instrument and are, therefore, described in detail in Chapter VI.

---

[13]Flanders, N.A., "Interaction Analysis in the Classroom: A Manual for Observers" (unpublished manuscript, University of Michigan, 1960(b)).

CHAPTER V

THE DESIGN OF THE STUDY

I. INTRODUCTION

This chapter describes the mechanics of the study. It includes an outline of the procedures followed in tape recording and transcribing the lessons, developing the instrument, and analyzing some lessons with the instrument.

II. DESCRIPTION OF THE DATA

Two series of tape recordings of classroom lessons were made which, for convenience, will be referred to as series A and series B. Series A consisted of two of each of Grade V, VIII, and IX mathematics lessons. All recordings were made at one elementary-junior high school in the city of Edmonton, Alberta. However, due to recording deficiencies it was impossible to use one of the Grade V lessons and one of the Grade IX lessons.

Series B consisted of three consecutive Grade IV mathematics lessons, two consecutive Grade VIII mathematics lessons and three consecutive Grade X mathematics lessons obtained from an elementary-junior high school and a high school in the city of Edmonton, Alberta.

Teachers were requested to carry on with normal textbook coverage. Review lessons were specifically excluded. The teachers were unaware of the nature of the investigator's

purpose for collecting these tapes.

Using a foot-controlled tape recorder, secretaries transcribed these tape recordings. Each new speaker was identified by S (student) or T (teacher). A margin was kept on the left-hand side of the page to facilitate the marking of units and their classification. If an utterance or part of one was unintelligible, it was indicated by "mumbled". The transcriptions were then checked by listening to the recordings, corrections being made where necessary.

The classroom recordings were made using a "mixer" which allowed for three microphones to be attached to the one recorder. This made it possible to distribute the microphones in the classroom for better reception. The details in all but the Grade X lessons of series B were handled by technicians. In the case of the latter, the investigator made the recordings.

III. THE PROCEDURE FOR DEVELOPING THE INSTRUMENT

The central problem in the development of an instrument is to identify objective characteristics of the concept being measured and then to identify the same characteristics in the data. Several stages in the process of developing an instrument were identified through the study of procedures used by various investigators.[1] The work of B. O. Smith

---

[1]Bellack, Arno A. in collaboration with Ronald T. Hyman, Frank L. Smith, Jr., and Herbert M. Kliebard. The

contributed most directly to the particular stages adopted here. His procedures proved to be useful since he too investigated the logical nature of the teaching process. His study has already been described in Chapter IV.

Two basic processes were dealt with. They are, in the words of Medley and Mitzel:

> Identifying a limited range of behavior relevant to the purpose of the study and . . . constructing categories or items to be used by the observer.[2]

The "limited range of behavior" is the unit and "categories or items" are the categories of the instrument. They are discussed in that order.

Language of the Classroom: Meanings Communicated in High School Teaching Phase Two (U.S. Office of Health, Welfare, and Education, Cooperative Research Project No. 2023, New York: Columbia University, 1965); Heynes, Roger W. and Ronald Lippit, "Systematic Observation Techniques," Handbook of Social Psychology, Gardner Lindzey, editor (Cambridge, Mass.: Addison-Wesley, 1954); Kliebard, Herbert Martin, "Teaching Cycles: A Study of the Pattern and Flow of Classroom Discourse" (Unpublished doctoral dissertation, Columbia University, 1963); Medley, D.M. and H.E. Mitzel, "Measuring Classroom Behavior by Systematic Observation," Handbook of Research on Teaching, N.L. Gage, editor (Chicago: Rand McNally, 1963); Meux, M. and B.O. Smith, "Logical Dimensions of Teaching Behavior," Contemporary Research on Teacher Effectiveness, B.J. Biddle and W.J. Ellena, editors (New York: Holt, Rinehart, and Winston, 1964); and Wright, Muriel J. and Virginia Proctor, "Systematic Observation of Verbal Interaction as a Method of Comparing Mathematics Lessons," U.S. Office of Education Cooperative Research Project No. 816 (St. Louis, Mo.: Washington University, 1961).

[2]Medley and Mitzel, *op. cit.*, p. 251.

<u>The Unit.</u> Prior to actually considering specific units, desirable characteristics of the prospective unit were identified. These dealt with such matters as size and boundaries. A number of units developed by other investigators were then examined to see how well they met those characteristics. Kliebard's "teaching cycle" was finally selected as the unit. Minor adaptations of the criteria developed by Bellack and Kliebard were made so as to make the unit more suitable for this study.

Finally, an estimate of the dependability of the criteria when used by different judges was obtained. Two judges studied the unit identification criteria and practised identifying units on three pages of transcriptions. The results were then compared and differences were noted and resolved by discussing the criteria. Each judge then proceeded to identify the units in 72 pages of transcriptions from the A series. The results were then compared by enumerating the number of units on which both judges agreed and the number of units identified by each judge. The formula for the coefficient of interjudge agreement used is based on a percentage agreement between two independent judgements: $R = \frac{A_{x,y}}{\max\ (E_x, E_y)}$ where $A_{x,y}$ is the number of agreements between judges x and y, E_x is the total number of units marked by x and E_y is the total number of units marked by y, and $\max\ (E_x, E_y)$ specifies that the larger of the two numbers E_x

and E_y is to be selected.[3]

The Categories. Medley and Mitzel have given several ideal characteristics of instruments for systematic observation of classroom behavior.[4] These along with a consideration of the intended use of the instrument were used to delineate the characteristics of the categories to be constructed. The actual choice of the categories and a description of each of them has already been given in Chapter II on pages 24 to 31. Using these categories and descriptions, a preliminary classification was made to determine the most useful part of the unit which would aid the classification. Particular efforts were made to see if the initiatory move of each unit would give sufficient information to carry out a classification.

To develop the unit classification critera, a trial-and-error method similar to that used in the B. O. Smith study[5] was used. The first step was to develop a preliminary set of criteria based on the definition of the categories. Then using these preliminary criteria, two judges worked independently classifying 427 units of verbal discourse. They made changes and added other criteria in order to

---

[3]Meux and Smith, op. cit., p. 138.

[4]Medley and Mitzel, op. cit.,p. 302.

[5]See Chapter IV, p. 47.

clarify ambiguities and make the criteria more practical. Having completed one such classification on 427 units, the judges consulted on their findings. Problems, agreements, and disagreements were noted. After deliberations, revisions were made jointly. The revised criteria were then used independently by each of the two judges on a further 679 units of discourse. Again the judges noted desirable changes and additions. Having completed this, final revisions of the criteria were made jointly by the two judges. These revised criteria were then designated as "the unit identification criteria."

Finally, an estimate of the dependability of these criteria when used by different judges was obtained. Two judges classified 628 units. The number of units placed in each of the categories by each of the judges was tabulated. Then the frequency of the number of agreements between the judges for each category was determined. This information was then used to determine a coefficient of interjudge agreement for each category by using the formula $R_i = \frac{A_i}{A_i + D_{1i} + D_{2i}}$ where A_i is the number of agreements between the two judges in category i, D_{1i} is the number of units placed in category i by the first judge but not by the second and D_{2i} is the number of units placed in category i by the second judge but not by the first.[6]

---

[6]Meux and Smith, op. cit., p. 150.

IV. ANALYSIS OF CLASSROOM DISCOURSE USING THE INSTRUMENT

To make a final check on the efficiency of the instrument and to demonstrate its intended use, the instrument was applied to the B series of lessons. The procedure for using the instrument is described in the Appendix. The results of the analysis are reported in Chapter VI and a suggested form for reporting the results is also given in the Appendix.

CHAPTER VI

THE RESULTS OF THE STUDY

I. INTRODUCTION

This chapter is a report on the decisions made by the investigator in each of the stages passed through during the process of developing the instrument. The report describes the alternatives considered and the reasons for the ultimate choices made. The investigator considered this information too important to be lost and hopes that its inclusion will give the reader a better understanding of the final format of the instrument. It is also the hope of the investigator that such information will encourage further attempts to deal with the same issue and will provide valuable guidance to such further investigation. The chapter also includes the results of the analysis of a series of lessons using the instrument.

II. REQUIREMENTS OF THE UNITS

To serve as a guide in the selection of a suitable unit, the following characteristics were considered and the requirements for each delineated:

1. Size. The unit had to be small enough so that at most one element of intuitive thinking would occur in it, but large enough so that an element of intuitive thinking could be recognized when present. Since the

elements of intuitive thinking describe thought processes, we were concerned here with having units which expressed, in a general sense, a "complete thought."

2. Boundaries. The boundaries of the unit had to be of such a shape that an element of intuitive thinking would occur completely within the unit, that is, an element of intuitive thinking should not be forced to "straddle" two units.

3. Focus. When a whole is broken up into parts, what is seen in the parts depends on the parts chosen. For example, when one is studying an essay and concentrating on the paragraphs, the focus would be on the ideas expressed in the essay. When the concentration shifts to the sentences, the focus shifts to matters of structure, style, grammar, etc. In making the choice of a unit for the purpose of this study, it was considered desirable to have a unit which would focus on, and even accentuate the thinking of the students and teacher.

4. Exhaustiveness. Since the instrument was to be valid in making comparisons from classroom to classroom and the length of the verbal discourses varied from classroom to classroom, the total verbal discourse had to be broken up into units. In this way a proportion of "intuitive units" to total units could be worked out for each classroom. Also, the units were required to be discrete rather than continuous.

5. Reliability. The units had to be identifiable in transcripts with a reasonable degree of reliability after a brief period of training in the use of the criteria. Use could not be made of audible or visual cues in helping to identify the units. A kind of unit was required that, ideally, would permit any analyst, trained to use the criteria, to identify the same units in any given transcript that any other analyst, equally trained, would find independently in the same transcript.

6. Neutrality. The unit had to be neutral with respect to teacher and grade level. Neutrality with repect to subject matter was not considered a necessary requirement, that is, the unit might be one which is appropriate to mathematics only.

III. CHOICE OF THE UNIT

Several units were considered before selecting the final one. Broadly speaking, in selecting a unit one has a choice between a time unit and a thought or natural unit.[1] Time units could have been selected to meet some of the requirements, for example, the requirement of reliability. However, in attempting to meet the requirements of boundaries the problem of elements of intuitive thinking

---

[1]Heynes, Roger W. and Ronald Lippit, "Systematic Observation Techniques," Handbook of Social Psychology, Gardner Lindzey, editor (Cambridge, Mass.: Addison-Wesley, 1954), p. 375.

straddling two consecutive units could not be eliminated. This handicap of the time unit was considered serious enough to be a cause for its rejection. This then meant that our unit would have to be a thought or natural unit.

Several possible units developed by other investigators were considered. Consideration was first given to the use of the episode and monologue breakdown developed by Smith and his associates and described in Chapter IV. The most serious difficulty encountered here was that of identifying episodes, that is, reliability. It was confusing to apply the criteria for distinguishing between a new aspect of the same topic and a clarification or amplification of the same point in the topic. Though Smith reported a reasonable degree of reliability, the nature of mathematics lessons makes it particularly troublesome. It was repeatedly found that teachers would branch out from topics and then return to the former discussion when they were dealing with sample problems. It was difficult to decide whether those branches were aspects of the main topic or not. If the solution were to take the whole discussion on a sample problem as the episode, then the unit becomes too large to be useful for purposes of this study. No other solution to the problem seemed promising.

Coombs, under the direction of Smith, developed a much larger unit which he called the strategy. This at first seemed to be a promising unit from the point of view of its definition: "the patterns of activities relevant to disclosing

content element."[2] The difficulty with this unit was practically the same as that with the episode. Strategies were thought of by Coombs as occurring in a further unit which he called a venture. "A venture is a unit of discourse consisting of a set of utterances dealing with a single topic and having a single over-arching objective."[3] The habit of mathematics teachers using sample problems to develop topics and using these as an occasion to make numerous "points" made it difficult to decide on a "single over-arching content objective." Again the requirement of reliability was not met.

The other unit considered and the one chosen for this study was Kliebard's "teaching cycle." This unit depends on four more basic units (pedagogical moves) developed by Bellack:[4] structuring, soliciting, responding, reacting. For purposes of the teaching cycle, structuring and soliciting are considered to be initiatory moves and responding and reacting are reflexive moves. A teaching cylce as defined by Kliebard and as used as a unit of verbal discourse in this

---

[2]Coombs, Jerrold Rex., "Teaching Strategies and the Teaching of Concepts," (Unpublished doctoral dissertation, University of Illinois, 1964), pp. 4-5.

[3]Ibid., p. 11.

[4]Bellack, Arno A, in collaboration with RonaldT. Hyman, Frank L. Smith, Jr., and Herbert M. Kliebard. The Language of the Classroom: Meanings Communicated in High School Teaching, Phase Two (U.S. Office of Health, Welfare, and Education, Cooperative Research Project No. 2023, New York: Columbia University, 1965).

study "is a unit of classroom [verbal] discourse which is initiated by a structuring move or a solicitation which is not preceded by a structuring and ending with the move that precedes a new structuring or a new unstructured solicitation."[5] Kliebard gives the following purpose for developing the teaching cycle:

> The teaching cycle is an attempt to conceptualize that larger unit of classroom discourse that seems to be inaugurated by Structuring and Soliciting Moves. It is an effort to see several Pedagogical Moves in combination in the hope that these patterns of Moves will provide a fruitful and reliable way of describing the ebb and flow of classroom discourse.[6]

To meet different needs several adaptations to the basic definition of Kliebard's teaching cycle were later developed by Bellack and associates.[7] Two of these seemed particularly appropriate to the purposes of this study. The first of these is the so-called "nodding solicitation." This refers to a type of solicitation made by classroom teachers in which they "give the nod" to the student before he may speak. This is often accomplished by calling the student's name, for example, "Mary?" Technically, this would initiate a new teaching cycle when it follows any other move than a structuring move, but for purposes of this study, such a move was

---

[5]Kliebard, Herbert Martin, "Teaching Cycles: A Study of the Pattern and Flow of Classroom Discourse" (Unpublished doctoral dissertation, Columbia University, 1963), p. 23.

[6]*Ibid.*, p. 15

[7]Bellack, *op. cit.*, Chapter 6.

incorporated into the teaching cycle initiated by the immediately preceding initiatory move.

The second adaptation deals with the situations where a repetition of an utterance or a part of an utterance is solicited. Again such a move would technically inaugurate a new teaching cycle, but for purposes of this study was incorporated into the teaching cycle initiated by the immediately preceding initiatory move.

In summary, the concept of teaching cycle as proposed for purposes of the instrument of this study is a portion of verbal discourse which begins with a structuring move or an unstructured solicitation and ends with the move immediately preceding a structured move or unstructured solicitation, except in cases of a (1) "nodding" solicitation and (2) a solicitation for repetition of an utterance or parts of an utterance, in which cases these solicitations are incorporated into the teaching cycle begun prior to this move.

IV. UNIT IDENTIFICATION CRITERIA

Since the unit chosen was one developed by another investigator, the criteria presented here have been largely adopted from Kliebard and Bellack, with minor modifications to make the unit more suitable for this study. The process of identifying teaching cycles requires two steps. The first step involves identifying the pedagogical moves as developed by Bellack and the second one involves a mechanical grouping

of the moves into teaching cycles.

Bellack's criteria for identifying the four pedagogical moves are as follows:

> Pedagogical Moves. Pedagogical moves, the basic units of classroom discourse, describe the verbal activities of teachers and pupils in the classroom. There are four basic types of moves which characterize the verbal interplay of teachers and pupils: structuring and soliciting, which are initiatory moves; and responding and reacting, which are reflexive moves.
>
> 1.1. Structuring (STR)
>
> Structuring moves serve the function of setting the context for subsequent behavior by (1) launching or halting-excluding interaction between pupils and teachers, and (2) indicating the nature of the interaction in terms of the dimensions of time, agent, activity, topic and cognitive process, regulations, reasons, and instructional aids. Structuring moves may set the context for the entire classroom game and/or for one or more subgames. As initiatory maneuvers, structuring moves are not called out by anything in the immediate classroom situation except the speaker's concept of what should be said or taught.
>
> Examples
>
> T/STR Last night I was sitting out on the porch trying to cool off and also trying to ask myself how to best begin this unit on international economics, international trade, world trade, or whatever else one wants to call it. And after thinking over a number of approaches, I wrote down on a paper a number of items that are connected with our story. Some of them are found in your pamphlet.
>
> T/STR All right, getting down to it now, I think international trade, then, or international economic relations, whatever you want to call it, is a field of study within economics which in many cases has been unfortunately divorced from domestic trade because there are great similarities, and also there are some rather distinct differences.
>
> . . .
>
> 1.2 Soliciting (SOL)
>
> Moves in this category are intended to elicit (a) an active verbal response on the part of the persons addressed; (b) a cognitive response, for example, encouraging persons addressed to attend to something; or (c) a physical response.
>
> Soliciting moves are clearly directive in intent and

function and are crucial in any active classroom interchange between teachers and pupils. Although these moves take all grammatical forms--declarative, interrogative, and imperative--the interrogative occurs most frequently.

Examples

T/SOL What are the factors of production?
P/SOL May we keep our books open?
T/SOL Turn the lights out, Bobby!
T/SOL Pay attention to this.

1.3 Responding (RES)

Responding moves bear a reciprocal relationship to soliciting moves and occur only in relation to them. This pedagogical function is to fulfill the expectations of soliciting moves and are, therefore, reflexive in nature. Since solicitations and responses are defined in terms of each other, there can be no solicitation that is not intended to elicit a response, and no response that has not been directly elicited by a solicitation.

Examples

T/SOL What are the factors of production?
P/RES Land, labor, and capital.
T/SOL Why didn't you do the assignment?
P/RES I was absent yesterday.
T/SOL What is exchange control?
P/RES I don't know.

1.4 Reacting (REA)

These moves are occasioned by a structuring, soliciting, responding, or another reacting move, but are not directly elicited by them. Their pedagogical function is to rate (positively or negatively) and/or to modify (by clarifying, synthesizing or expanding) what was said in the move (s) that occasioned them. Reacting moves differ from responding moves, in that while a responding move is always directly elicited by a solicitation, preceding moves serve only as the occasion for reactions. For example, rating by a teacher of a student's response is designated a reacting move; that is, the student's response is the occasion for the teacher's evaluative reaction, but it does not actively elicit it.

Example

T/REA That's partly it.
P/REA But he left out the most important part.
T/REA Good. It limits specifically the number of items of one type or another which can come into this country. For example, we might decide that no more than one thousand of German Automobiles will be imported in any one calendar year. This is a specific quota which the government checks.[8]

---

[8]Kliebard, op. cit., pp. 194-196.

To facilitate the final identification of the teaching cycle units, one modification was made. The "nodding" solicitation and the "repeating" solicitation were coded NOD SOL and REPEAT SOL, respectively. This was done so as to identify those solicitations which were definitely not to be considered as initiatory moves. For example, we would have the following units coded thus:

SOL/T Is 91 divisible by any of these factors: 2,3,5?
NOD SOL/T Doug?
REPEAT SOL/T Would you speak up please.
REPEAT SOL/T Pardon me, please.

The criteria, developed by Kliebard, for identifying teaching cycles were found to be a mechanical triviality after the pedagogical moves had been identified. They were inherent in the definition of teaching cycle presented earlier and required the identification of structuring moves or unstructured soliciting moves to indicate both the beginning of a new unit and the ending of the previous one. The mechanical procedure used was to draw vertical square brackets from the first structuring move or unstructured soliciting move to the move preceding the next structuring move or unstructured soliciting move. For example:

SOL/T Why?
RES/S A 2 won't go into it.
REA/T 2 won't go into 111.
SOL?T Why?

SOL/T This fraction then reduces to?
NOD SOL/T Graham?
RES/S 2/3
REPEAT SOL/T Speak up,Graham.
RES/S 2/3
REA/T 2/3, right.

[STR/T Now here we concentrated on 74. We could just as well have concentrated on the 11. Let's take a look at 74 over 111.

A complete bracket identified a single teaching cycle unit, in this study referred to as a single unit of verbal discourse.

V. RELIABILITY OF UNIT IDENTIFICATION CRITERIA

A summary of the results of applying the unit identification criteria to the transcriptions of the A series by two independent judges and their resulting reliability coefficients is given in Table I. (The formula used is discussed in Chapter V.)

TABLE I

RELIABILITY OF THE UNIT IDENTIFICATION CRITERIA

CLASS	Pages of transcriptions	Total no. of units identified by Judge 1	Total no. of units identified by Judge 2	No. of agreements	Reliability coefficient
IX-1	15.5	137	137	111	.81
VIII-1	13.5	89	93	42	.45
VIII-2	10.5	76	79	46	.58
VI-1	17.5	130	92	67	.52
VI-2	10	83	80	64	.77
V-1	8.5	69	69	45	.65
Total	75.5	584	550	375	.64

VI. REQUIREMENTS OF THE CATEGORIES

In setting out the minimum requirements of the categories, the intended use of the instrument and the ideal characteristics of instruments for systematic observation as conceived by Medley and Mitzel[9] served as guide-posts. The following requirements of the categories were established:

1. The categories could not overlap, but were to be discrete. That is, one category was required to define behavior which was distinguishable from the behavior defined by another category.
2. There was to be a category into which each unit of a verbal discourse could be classified. This meant that an "other" category was required to take care of all non-intuitive thinking units and enough categories so that each unit containing an element of intuitive thinking could be placed into a category.
3. Those categories defining elements of intuitive thinking were required to represent a positive, singular, present tense occurrence of the behavior.
4. The categories were to represent behavior which was observable and which was definable in simple, concrete terms.

---

[9] Medley, D.M. and H.E. Mitzel, "Measuring Classroom Behavior by Systematic Observation," Handbook of Research on Teaching, N.L. Gage, editor (Chicago: Rand McNally, 1963).

VII. DEVELOPING THE CATEGORIES

The following categories were selected on the basis of the requirements set forth in the preceding section and on the basis of the list of elements of intuitive thinking selected and defined in Chapter II:

Category 1. Symmetry (SYM)

Category 2. Analogy (ANA)

Category 3. Similarity (SIM)

Category 4. Generalization (GEN)

Category 5. Experimentation (EXP)

Category 6. Manipulation (MAN)

Category 7. Skipping Steps (SKI)

Category 8. Preverbalization (PRE)

Category 9. Solution visualization (SOL)

Category 10. Informal proof (INF)

Category 11. Other (OTH)

The reasons for selecting these particular categories have already been presented in the last section of Chapter II dealing with the behavioral definition of intuitive thinking.

VIII. THE PART OF THE UNIT TO BE USED IN CLASSIFYING UNITS

It is the function of the initiatory moves to determine the direction of thought for the whole unit. The only other moves which could sustain a unit are the responding and reacting moves and both are dependent, directly or indirectly, on the initiatory moves, as can be seen by an examination of

their definitions. On the basis of trial and error attempts at classification and the fact that some units may only consist of a single initiatory move, the part of the unit chosen to be used in classifying the units was the initiatory move, that is, a structuring move or an unstructured solicitation. However, it was sometimes necessary to refer to the following moves to determine the intent of the initiatory move.

IX. UNIT CLASSIFICATION CRITERIA

This particular stage in the development proved to be the most difficult and, at the same time, the most unproductive. The original criteria were prepared directly from a consideration of the definition of the categories (as already reported in Chapter II). The conferences with the judges after the preliminary classification attempts were unproductive as far as adding criteria beyond those directly implied by the definition. It was discovered that most differences were easily reconciled after reasons were verbally given for the particular classification. This suggests that teams of at least two judges each may have produced more significant results.

In most cases, the discussions pointed out inadequacies in the judges' understanding of the categories. This led to the inclusion of practice units in the final instrument.[10]

---

[10]See the Appendix.

The following are the final definitions and criteria accepted for the categories:[11]

1. Symmetry is the property of a mathematical problem which shows that one or more of the following exist in the problem: parts may be interchanged, there may be repeating patterns, or there may be a correspondence between different parts of the problem.

Criteria:

a) Attention is drawn to the pattern in a structure or problem.

b) Attention is drawn to the parts which may be interchanged.

c) Parts of formal proofs where the phrase "similarly this can be proved" are classified as symmetry.

2. Analogy is a procedure in which a comparison between a mathematics problem or structure and a physical embodiment or illustration is made.

Criteria:

a) An embodiment of a mathematical structure in concrete, structured or physical materials is made.

b) An embodiment of a mathematics structure in a Dienes' type of mathematical story is made.

c) An embodiment of a mathematical structure in a Dienes' type of mathematical game is made.

---

[11]Examples illustrating each of the categories are given in the Appendix.

d) A physical illustration of an abstract concept is given.

3. Similarity is a procedure in which a comparison is made between two mathematics structures or mathematics problems, or parts thereof, having elements in common for the purpose of enlightenment of the present problem or structure.

Criteria:

a) Recall of a similar or identical problem and procedures used there is requested or given.

b) Recall of a similar mathematical structure and the techniques associated with that structure is made.

c) Straight recall for the purpose of review is not included.

d) Recall for the purpose of emphasizing contrasts and differences to a present situation are included.

4. Generalization is the procedure of examining specific facts, examples or cases and seeking generalizations of principles and techniques.

Criteria:

a) Studying a specific case and finding a general rule.

b) Studying a specific relationship and discovering a general proof.

5. Experimentation is the serious "grcping" for or

trying out of a plausible answer or procedure.

Criteria:

a) Calling for guesses or approximations to the solution.

b) Serious student responses with a large degree of uncertainty are given.

6. Manipulation is the activity of manipulating or otherwise examining the observable characteristics of the information given with the view to becoming acquainted with the elements of the problem and understanding their meaning, implications and relationships. There is a concentration on the given information rather than on possible solutions.

Criteria:

a) Examining the inter-relationships among the various elements or parts of a mathematical structure.

b) Comparing the various characteristics of the elements of a problem or structure.

c) Familiarizing the participants with the elements of the problem with complete disregard for the relevance to the solution.

d) It is not a maneuvering of the elements in order to gain a solution.

7. Skipping steps is a procedure for solving a problem or conceptualizing a mathematical structure in which

many of the steps of the argument expected at a certain level of understanding are apparently omitted.

Criteria:

a) Omitting logical steps in the solution of a problem or in conceptualizing a mathematical structure.

b) The cue as to whether or not steps have been omitted for a certain level of understanding is often found in the following units.

8. Pre-verbalization is the use of concepts, principles, or rules without verbalizing them.

Criteria:

a) Rules are used without stating them in words.

b) A concept or principle is used without the user being conscious or aware of its use.

c) The user expresses certainty regarding the concept, but expresses ignorance as to why or how it is true.

9. Solution visualization is a pre-occupation with the nature of the solution or finished structure for the purpose of delimiting the possible magnitude, shape, position, or other characteristics of the solution or structure.

Criteria:

a) Some characteristics of the finished structure or solution are delineated.

b) An actual solution or finished structure is not

necessarily designated.

c) When an actual solution is designated, the emphasis is on its characteristics.

d) Parameters of the solution are outlined.

e) Solution visualization is to the solution what manipulation is to the given information.

10. Informal proof is a procedure whereby visual or "inductive" arguments which are satisfying to the student are substituted for a formal proof.

Criteria:

a) A visual proof, such as a picture, is cited.

b) An example is given to show the validity of a rule, concept, etc.

c) A series of special cases or examples or illustrations are given to exemplify the validity of generalization.

X. RELIABILITY OF THE UNIT CLASSIFICATION CRITERIA

A summary of the results of applying the unit classification criteria to the transcriptions of the B series by two judges is given in Table II. The table shows the number of units in each lesson placed in each of the categories and the reliability (inter-judge agreement) of each of the categories and the mean reliability. (The formula used is given in Chapter V.)

TABLE II

RELIABILITY OF UNIT CLASSIFICATION CRITERIA

CATEGORY	CLASS																		Number of agreements between J_1 and J_2	Reliability
	X-1		X-2		X-3		VIII-1		VIII-2		IV-1		IV-2		IV-3		Total			
	J_1*	J_2**	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2		
SYM	3	5	1	0	1	1	0	0	0	0	0	0	0	0	0	0	5	6	5	.83
ANA	0	0	2	0	4	1	1	0	0	0	7	4	0	10	1	0	15	15	2	.07
SIM	4	0	1	0	4	2	4	7	3	3	0	0	2	4	1	15	19	31	5	.11
GEN	0	0	0	0	0	0	6	5	3	0	0	0	0	0	1	1	10	6	4	.33
EXP	2	5	3	0	2	4	0	1	0	0	1	0	1	1	0	0	9	11	2	.11
MAN	3	4	0	5	0	1	18	1	3	5	13	9	9	2	21	2	67	29	10	.12
SKI	0	0	0	0	0	0	0	0	0	0	0	1	0	2	1	1	1	4	1	.75
PRE	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-
SOL	0	0	0	0	1	0	0	1	4	13	0	0	0	0	0	3	5	17	3	.16
INF	2	2	0	0	2	3	0	1	1	0	9	4	0	1	0	0	14	11	7	.39
Total	14	16	7	5	14	12	29	16	14	21	30	18	12	20	25	22	145	130	39	.17
Total no. of units identified	90		31		47		89		14		90		70		97					
Pages of Transcript	16.5		6.5		11		10.5		13.5		10.5		7.5		10.5					

*No. of units placed in each category by Judge 1.

**No. of units placed in each category by Judge 2.

XII. RESULTS OF THE CLASSROOM ANALYSIS

A summary of the results of applying the instrument to the lessons of the B series is given in Table III.

TABLE III

CLASSIFICATION OF THE UNITS OF THE B SERIES LESSONS BY CATEGORIES

Class-room	Total No. of Units	SYM	ANA	SIM	GEN	EXP	MAN	SKI	PRE	SOL	INF
X	168	5	6	9	0	7	3	0	0	1	4
VIII	203	0	1	7	9	0	21	0	0	4	1
IV	257	0	8	3	1	2	43	1	0	0	9

Table IV shows the "intuitive ratio" for each classroom. The "intuitive ratio" is the ratio of total number of units placed in a category to the total number of units identified in the lessons.

TABLE IV

INTUITIVE RATIO BY CLASSROOM

Class-room	Total No. of Units	No. of Intuitive Units	Intuitive Ratio
X	168	35	.21
VIII	203	43	.21
IV	257	67	.26

On the basis of these results, Classroom X and Classroom VIII would each receive equivalent ranks. That is, both Classroom X and Classroom VIII use intuitive thinking to the same extent. Classroom IV would receive a higher rank, indicating that intuitive thinking is used to a larger extent in this classroom than was the case in the other two classrooms.

In the particular lessons used in this study, symmetry was used exclusively at the Grade X level whereas analogy was used most frequently at the Grade IV level. Manipulation was noticeably more common in Grade VIII and Grade IV than at the Grade X level. The other elements of intuitive thinking showed no significant trends.

Symmetry had the best inter-judge agreement, the reliability coefficient being .83. Skipping steps had a reliability coefficient of .75. However, one judge used this category only once, while the other used it four times. The informal proof and generalization categories had reliability coefficients of .39 and .33, respectively. All other categories had reliability coefficients below the .20 level. These low levels are, in part, attributable to inadequate training of the judges.

CHAPTER VII

CONCLUSIONS AND IMPLICATIONS

I. INTRODUCTION

This chapter includes some general observations and a discussion of problems associated with the present instrument. Possible uses of the instrument are enumerated and implications for classroom teachers and for future research are identified.

II. CONCLUSIONS

The theoretical role of intuitive thinking, as presented in Chapter II and III, has demonstrated that intuitive thinking can be integrated into established theories of mathematics learning. The role of intuitive thinking has been presented as complementing analytic thinking. Its chief function was described as being that of alleviating the poverty of mathematical imagery in most children's minds. It was pointed out that employing intuitive thinking gives the learner a wide range of meaningful mathematical imagery so necessary in the early stages of concept development.

The instrument developed consisted of two parts. The transcripts of lessons were broken down into the four moves of Bellack: structuring, soliciting, responding, and reacting. These were then grouped into Kliebard's teaching cycle.

These units were then classified as being one of the following elements of intuitive thinking: symmetry, analogy, similarity, generalization, experimentation, manipulation, skipping steps, preverbalization, solution visualization, or informal proof. If a unit did not meet the criteria of one of these categories, it was classified as non-intuitive.

There was a low incidence of elements of intuitive thinking in the lessons used in this study. In an analysis of 628 units, only 145 were classified as intuitive by one judge and 130 by another judge. The more significant observation, however, was that not a single unit of preverbalization was identified by either judge. Only one unit of skipping steps was identified by one judge and four units by the other judge.

In spite of the somewhat limited success of the present study, the investigator feels the exercise was useful in that a definite step has been taken towards developing a more concrete understanding of intuitive thinking and its relationship to effective mathematics teaching.

III. GENERAL OBSERVATIONS

Although the major purpose of the study was to develop the instrument, the investigator was also able to make several interesting observations during this process. One of these observations was that of the ten elements of intuitive thinking, no more than six were found in one

lesson, and often two or three would predominate in the lesson.

There was very little occurrence of the element "skipping steps" and a complete absence of the element "preverbalization." The reason for this could be that the criteria for detecting these are inadequate or that they are of such a nature that they are hidden in the lesson. Or, perhaps, teachers are simply not making use of these elements. In fact, the investigator was constantly frustrated by the scarcity of all of the elements of intuitive thinking in the lessons used. It is the opinion of this investigator that the teachers dealt with in this study were either completely unaware of the concept of preverbalization or else considered it completely inappropriate as a teaching technique. As well, the elements "symmetry" and "solution visualization" rarely occurred in the lessons. In no cases did the investigator feel he was in contact with a lesson which exhibited a significant emphasis on an intuitive approach.

At the same time, the investigator found that several writers on the subject of teaching methodology considered the use of an intuitive approach essential for an adequate understanding of mathematical concepts. Dienes, Bruner and Evans are very emphatic on this point. Others dealt with specific elements of intuitive thinking. Gertrude Hendrix, for example, is an ardent proponent of the use of preverbalization. Polya stresses the use of skipping steps, analogy,

generalization, informal proof and experimentation.

IV. SOME PROBLEMS WITH THE PRESENT INSTRUMENT

One of the problems encountered during the analysis of the B series lessons had to do with the size of the unit. The unit had originally been worked out on arithmetic and algebra lessons. The grade X lessons of the B series were a sequence of geometry lessons. These proved to be more difficult to classify. In many cases the unit appeared to be too large because often more than one element of intuitive thinking was present in each unit. However, the problem may not be with the size of unit, but with the criteria which had been developed on lessons with no geometry content.

A second problem with the instrument is its low reliability (inter-judge agreement) in the unit classification phase. This, of course, limits the reliability of the instrument as a whole. However, since the purpose of the instrument is to <u>rank-order</u> classrooms, and not to identify specific elements of intuitive thinking, a true measure of the reliability of the instrument would have to compare the rank-orders of several classrooms achieved by a number of different judges. The reliability of the instrument might improve when considered from this point of view.

Another consideration with respect to the reliability of the classification criteria has to do with the training of the judges. During the developmental phase of determin-

ing the criteria, differences in classification were usually resolved readily when the judges explained their reasons for a certain classification. The same thing was evident in the determination of the reliability of the criteria. In one particular case--that of manipulation--it was discovered that there was a major difference in the understanding of what was meant by the criteria. It is evident that the training of the judges was inadequate.

One aspect of the instrument which causes an inconvenience is the time required to analyze lessons. To analyze a forty minute lesson would require an estimated time of one and one-half to two hours. However, since much of this time is listening and reading time, it does not seem possible to significantly reduce this time. Secretarial help can be used for one hour of that time, namely, for transcribing the lessons.

V. IMPLICATIONS FOR TEACHERS

If some of the general observations made about the lessons used in this study can be generalized and if the theoreticians are right (at least some of them have experimental evidence to back them up, namely Dienes and Hendrix), teachers will need to take note of these elements of intuitive thinking if they wish to improve the quality of mathematics teaching. On the basis of the review of the literature, the investigator is convinced that there is much value

in the use of an intuitive approach at all levels of school mathematics teaching.

The instrument could provide teachers with useful feedback on the kinds of things they are actually doing in their classrooms. The teacher could identify the particular kinds of intuitive elements she uses extensively and those she uses only occasionally, or not at all. Such information could assist teachers in making certain adjustments in teaching methods.

VI. IMPLICATIONS FOR FUTURE RESEARCH

The original stimulus for the present study came from an attempt to determine the present status of an intuitive approach in mathematics classrooms. However, no instrument related to intuitive thinking was available for making such measurements. This instrument could now be used to determine the present status of an intuitive approach.

The instrument could be useful in determining the extent to which training can influence a teacher's use of an intuitive approach. It would also be a means of determining the relationship between an intuitive approach and student learning.

It could be used to determine if certain elements of intuitive thinking are more frequently used at certain grade levels than at others and whether or not certain teachers make use of specific elements more than other

teachers.

This an exploratory study. Similar studies are urgently needed for comparative purposes. Independent research must be continued to further define what is meant by an intuitive approach. Are there some elements included in this study which are not of an intuitive nature? Have other elements been omitted which should have been included? Is the theoretical basis presented here sound?

The modest success of the present study demonstrates that the concept of intuition can be dealt with concretely and quantitatively. Certain problems with the methodology of this study need to be corrected. The present investigator discovered that the lessons used did not have a sufficient number of units illustrating each of the categories on the instrument. There was insufficient experience material to construct adequate criteria for classifying units. This often led to the temptation to change the definition of a category to something for which examples were available. It is felt that the development of the instrument would have been more successful had the teachers been given specific instructions to incorporate some of the elements of intuitive thinking into their lessons.

The procedure used for developing the instrument in this study was effective. By using data that has more elements of intuitive thinking, more precise statements of the criteria for classifying the units could be obtained by using

the same procedure.

As already indicated, the size of the unit selected for the instrument was not entirely satisfactory. Also, the identification of the unit requires a considerable fraction of the total time needed to make the measurement. Are there other units available which could be used with the categories of the present instrument?

The categories should be studied in conjunction with actual data to see if some of them occur together. Can they perhaps be dimensionalized? Does a high frequency of a particular category always accompany a low frequency of another category?

The complete absence of preverbalization in the lessons analyzed in this study prompts several questions. Are teachers aware of the concept? Is preverbalization quantifiable and observable? Can it occur side by side with other elements of intuitive thinking?

BIBLIOGRAPHY

BIBLIOGRAPHY

A. BOOKS

Association for Supervision and Curriculum Development. New Curriculum Developments. Washington, D.C.: Association for Supervision and Curriculum Development, 1965.

_______. The Way Teaching Is. Washington, D.C.: Association for Supervision and Curriculum Development, 1966.

_______. Theories of Instruction. Washington, D.C.: Association for Supervision and Curriculum Development, 1965.

Ausubel, D.P. The Psychology of Meaningful Verbal Learning. New York: Grune and Stratton, 1963.

Bellack, Arno A. The Language of the Classroom. New York: Teachers College Press, Teachers College, Columbia University, 1966.

Bruner, Jerome, S. On Knowing. Cambridge: Belknap Press of Harvard University Press, 1962.

_______. The Process of Education. New York: Vintage Books, 1960.

Burden, Virginia. The Process of Intuition. New York: Greenhich Book Publishers, 1957.

Burton, W.H., R.B. Kimball and R.L. Wing. Education for Effective Thinking. New York: Appleton-Century-Crofts, Inc., 1960.

Dienes, Z.P. An Experimental Study of Mathematics Learning. London: Hutchinson Educational, 1963.

_______. Building Up Mathematics. London: Hutchinson Educational, 1960.

_______. Concept Formation and Personality. London: Leicester University Press, 1965.

_______. Mathematics and the Primary School. London: MacMillan, 1964.

_______. Modern Mathematics for Young Children. London: Educational Supply Association, 1965.

_______. The Power of Mathematics. London: Hutchinson Educational, 1964.

Ewing, A.C. Reason and Intuition. Annual Philosophical Lecture, Henriette Hertz Trust, British Academy, 1941. From the proceedings of the British Academy, Volume XXVII. London: Humphrey Milford Amen House, E.C.

Guilford, J.P., P.R. Merrifield, and A.B. Cox. Creative Thinking in Children at the Junior High School Levels. Cooperative Research Project No. 737, Reports from the Psychological Laboratory, No. 26. Los Angeles: University of Southern California, September, 1961.

Hadamard, J.S. An Essay on the Psychology of Invention in the Mathematical Field. New York: Dover Publications, 1954.

International Study Group for Mathematics Learning. Mathematics in Primary Education. Hamburg: Unesco Institute for Education, 1966.

Isaacs, Nathan. New Light on Children's Ideas of Numbers: The Work of Professor Piaget. London: Educational Supply Association, 1960.

Lovell, K. The Growth of Basic Mathematical and Scientific Concepts in Children. London: University of London Press, 1962.

Nidditch, P.H. Elementary Logic of Science and Mathematics. Glencoe, Ill.: The Free Press of Glencoe, 1960.

Piaget, J. The Psychology of Intelligence. Patterson, New Jersey: Littlefield, Adams and Co., 1960.

Polya, G. How to Solve It. Princeton, New Jersey: Princeton University Press, 1957.

_______. Mathematical Discovery. New York: John Wiley and Sons, 1962.

_______. Mathematics and Plausible Reasoning. Princeton, New Jersey: Princeton University Press, 1954.

Popper, K.R. The Logic of Scientific Discovery. New York: Basic Books, 1959.

Sarbin, T.R., R. Taft, and D.E. Bailey. Clinical Inference and Cognitive Theory. New York: Holt, Rinehart, and Winston, 1960.

Smith, B.O. and others. A Tentative Report on the Strategies of Teaching. U.S. Department of Health, Education and Welfare, Office of Education, Cooperative Research Project No. 1640. Urbana: Bureau of Educational Research, College of Education, University of Illinois, 1964.

Willoughby, S.S. Contemporary Teaching of Secondary School Mathematics. New York: Wiley, 1967.

B. ARTICLES IN COLLECTIONS

Ashner, Mary J. "The Language of the Teaching," Language and Concepts in Education, B.O. Smith and Robert H. Ennis, editors. Chicago: Rand McNally and Company, 1961.

Crawford, D.H. "Stages of Intellectual Development--Their Meaning for Mathematics Courses and Methods," New Thinking in School Mathematics, pp. 113-114. Report of a seminar held by the Canadian Teachers' Federation, at Ottawa, April 29-30, 1960.

Dienes Z.P. "Research in Progress," New Approaches to Mathematics Teaching, F.W. Land, editor. London: MacMillan, 1963, pp. 49-57.

Getzels, J.W. "Creative Thinking, Problem-solving and Instruction," Theories of Learning and Instruction, Sixty-third Yearbook of the National Society for the Study of Education. Chicago: National Society for the Study of Education, 1964. pp. 240-268.

Guilford, J.P. "Basic Problems in Teaching for Creativity," Instructional Media and Creativity, C.W. Taylor and F.E. Williams, editors. New York: Wiley, 1966. pp.71-103.

Heyns, Roger W. and Ronald Lippitt. "Systematic Observational Techniques," Handbook of Social Psychology, Gardner Lindzey, editor. Cambridge, Mass.: Addison-Wesley, 1954. pp. 370-404.

Hildebrandt, E.H.C. "Mathematical Modes of Thought," The Growth of Mathematical Ideas, K-12, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: National Council of Teachers of Mathematics, 1959.

Lankford, F.G., Jr. "Implications of the Psychology of Learning for the Teaching of Mathematics," The Growth of Mathematical Ideas, K-12, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: National Council of Teachers of Mathematics, 1959.

Medeley, D.M. and H.E. Mitzel. "Measuring Classroom Behavior by Systematic Observation," Handbook of Research on Teaching. N.L. Gage, editor. Chicago: Rand McNally, 1963.

Meux, M. and B.O. Smith. "Logical Dimension of Teaching Behavior," Contemporary Research on Teaching Effectiveness, Bruce, J. Biddle and William J. Ellena, editors. New York: Holt, Rinehart and Winston, 1964. pp. 127-164.

National Council of Teachers of Mathematics. Research in Mathematics Education. Washington, D.C.: National Council of Teachers of Mathematics, 1967.

Skemp, R.R. "A Three-part Theory of Learning Mathematics," New Approaches to Mathematics Teaching, F.W. Land, editor. London: MacMillan, 1963. pp. 40-48.

Smith, B.O. "A Concept of Teaching," Language and Concepts in Education, B.O. Smith and Robert H. Ennis, editors. Chicago: Rand McNally and Co., 1961.

Stein, Harry L. "Implications of Theories of Learning for School Mathematics," New Thinking In School Mathematics, pp. 99-112. Report of a seminar held by the Canadian Teachers' Federation at Ottawa, April 28-30, 1960.

Westcott, M.R. "Empirical Studies of Intuition," Widening Horizons of Creativity, C.W. Taylor, editor. New York: Wiley, 1964. pp. 34-53.

C. ARTICLES IN PERIODICALS

Adler, Irving. "Mental Growth and the Art of Teaching," The Mathematics Teacher, LIX (December, 1966), 706-715.

Adler, Marilynee J. "Some Educational Implications of the Theories of Jean Piaget and J.S. Bruner," Canadian Education and Research Digest, IV (December, 1964), 291-305.

Biggs, J.S. "Towards a Psychology of Educative Learning," International Review of Education, XI (January, 1965), 77-93.

_______. "The Development of Number Concepts in Young Children," Educational Leadership, I (February, 1959), 17-34.

Brune, I.H. "Geometry in the Grades," The Arithmetic Teacher, VIII (May, 1961), 210-219.

Buswell, G.T. "Educational Theory and the Psychology of Learning," The Journal of Educational Psychology, XLVIII (March, 1956), 175-184.

Dienes, Z.P. "The Growth of Mathematical Concepts in Children Through Experience," Educational Research, II (November, 1959), 19-28.

Hendrix, Gertrude. "Learning by Discovery," The Mathematics Teacher, LIV (May, 1961), 290-299.

_______. "A New Clue to Transfer of Training," Elementary School Journal, XLVIII (December, 1947), 197-208.

Hohn, F.E. "Teaching Creativity in Mathematics," The Arithmetic Teacher, VII (March, 1961), 102-106.

Kemeny, J.G. "Rigor vs. Intuition in Mathematics," The Mathematics Teacher, XIV (February, 1961), 66-74.

Skemp, R.R. "Reflective Intelligence and Mathematics," British Journal of Educational Psychology, XXXI (February, 1961), 45-55.

_______. "The Need for a Schematic Learning Theory," British Journal of Educational Psychology, XXXII (February, 1965), 133-142.

Westcott, M.R. "On the Measurement of Intuitive Leaps," Psychological Reports, IX (1961), 267-274.

Wright, Muriel J. "Development of an Instrument for Studying Verbal Behavior in a Secondary Mathematics Classroom," Journal of Experimental Education, XXVIII (1959), 103-121.

D. UNPUBLISHED MATERIALS

Ashner, Mary J. "The Analysis of Classroom Discourse: A Method and Its Uses." Unpublished doctoral dissertation, University of Illinois, 1958.

Bellack, Arno A. in collaboration with Ronald T. Hyman, Frank L. Smith, Jr., and Herbert M. Kliebard. "The Language of the Classroom: Meanings Communicated in High School Teaching," U.S. Department of Health, Education and Welfare, Office of Education, Cooperative Research Project No. 2023. New York: Institute of Psychological Research, Teachers College, Columbia University, 1965.

Coombs, Jerrold Rex. "Teaching Strategies and the Teaching of Concepts." Unpublished doctoral dissertation, University of Illinois, 1964.

Evans, E.W. "Measuring the Ability of Students to Respond in Creative Mathematical Situations at the Late Elementary and Junior High School Level." Unpublished doctoral dissertation, University of Michigan, 1964.

Flanders, N.A. "Interaction Analysis in the Classroom: A Manual for Observers." Unpublished manuscript, University of Michigan, 1960 (b).

Kliebard, Herbert Martin. "Teaching Cycles: A Study of the Pattern and Flow of Classroom Discourse," Unpublished doctoral dissertation, Columbia University, 1963.

Smith, B.O. and Milton O. Meux, in collaboration with Jerrold Coombs, Daniael Eierdam and Ronald Szoke. "A Study of the Logic of Teaching." U.S. Department of Health, Education, and Welfare, Office of Education, Cooperative Research Project No. 258(7257). Urbana: Bureau of Educational Research, College of Education, University of Illinois, 1962.

Taba, Hilda, Samuel Levine, and Freeman F. Elzey. "Thinking in Elementary School Children." U.S. Department of Health, Welfare, and Education, Office of Education, Cooperative Research Project No. 1574. San Francisco: San Francisco State College, 1964.

Wright, Muriel J. and Virginia Proctor. "Systematic Observation of Verbal Interaction as a Method of Comparing Mathematics Lessons," U.S. Office of Education Cooperative Research Project No. 816. St. Louis, Mo.: Washington University, 1961.

APPENDIX

THE INSTRUMENT

The purpose of this section is to present suggestions on how to learn to use the instrument and how to record results obtained with the instrument.

By intuitive approach is meant a teaching approach in which the use of intuitive thinking is encouraged. Intuitive thinking is used as it was defined on page 32 of Chapter II. This instrument has been developed in conjunction with studies of mathematics lessons and its categories are not intended to be descriptive of other school subject areas. For this reason it is also essential that the user of this instrument be acquainted with the mathematical concepts being developed in the lessons he is analyzing.

The instrument's use also requires the user to commit to memory various criteria. To aid the prospective user, practice materials with keys are included in following sections of this Appendix.

The purpose of the instrument is to rank-order classrooms on the basis of the extent to which they make use of intuitive thinking during a mathematics teaching-learning situation. The measurement points out <u>variations</u> from classroom to classroom; it does not give a measure of the "amount" of intuitive thinking engaged in by members of the classroom.

A Suggested Procedure to be Followed in the Use of the Instrument

1. Tape the classroom lessons. With the use of a tape recorder equipped with a device to feed in several microphones, record the desired classroom lessons. Be sure to use a coding system for later identification of the tapes.

2. Transcribe the lessons. With the use of a foot-pedal control tape recorder or a dictaphone, transcribe the lessons. Mark each new speaker by T (teacher) or S (student). Leave a two and one-half inch margin on the left-hand side of the page for identifying and classifying units. Make a final review of the transcriptions by checking them against the tapes.

3. Study the unit identification criteria. This step may be considered in two phases. Phase one requires a thorough study of the criteria for identifying each of the moves: STR, SOL, NOD SOL, REPEAT SOL, RES, and REA. These criteria are presented on pages 70-71 of Chapter VI.

Phase two requires a study of the definition of the unit (teaching cycle). This is presented on pages 72 to 73 of Chapter VI.

4. Practice unit identification. Three pages of transcriptions with units already identified are given on pages 105 to 108 of this Appendix. Study these to see how the moves are identified and how the units are identified. Note the mechanics of the procedure: the slash for dividing

moves, the names of the moves in the margin, and the use of the square bracket to group the moves into the units. A further three pages of lessons are given on pages 109 to 111 of this Appendix. Identify the units in these lessons and check the results with those given on pages 112 to 114 .

5. Identify the units in your lessons. Using the procedure outlined in the preceding paragraph, proceed to identify the units in your raw data. A preliminary reading of the whole lesson should precede the identification of the units.

6. Study the unit classification categories and their criteria. These are found on pages 76 to 81 of Chapter VI. Background reading on these categories is given on pages 23 to 32 of Chapter II. This step in the procedure requires close scrutiny and careful study. Familiarity is essential so that the characteristics of the categories will be recognized in the units as they are read.

7. Practice the unit classification. Study the examples of the units and their classification presented on pages 105 to 108 of this Appendix.

Reporting the Results

The results for each classroom, or for a specific teacher can be computed as an "intuitive ratio" by enumerating the total number of units identified and the total number of these units which were classified into at least one of the

categories. The "intuitive ratio" is the ratio of the number of classified units to the total number of identified units. These ratios are then arranged in numerical order, thus giving the rank-ordering of the classrooms on the basis of the extent to which they made use of intuitive thinking during a mathematics teaching-learning situation.

EXAMPLES OF UNITS

The following transcription is intended to illustrate

(1) Bellack's four moves: structuring (STR), soliciting (SOL), responding (RES), reacting (REA)

(2) the grouping of the moves into units (Kliebard's teaching cycle)

(3) the mechanics of identifying the units.

Each square bracket identifies one unit.

STR SOL NOD SOL	T	In the last lesson we talked about our numeral system. We talked about the fact that it was based on tens. Now before I take a look at another numeral system, I would like us to just quickly review the principles that we discussed or decided upon as far as our numeral system is concerned./ Who can tell me the first principle that we discussed?/ (name)./
RES	S	That our numeral system has ten digits./
REPEAT SOL	T	Speak up please./
RES	S	Our numeral system has ten digits./
SOL	T	And quickly, what are these digits?/
RES	S	1, 2, 3, 4, 5, 6, 7, 8, 9, 0.
REA SOL	T	Alright, that's one point./ Now the second principle that we discussed./
RES	S	It's - that it's made up of tens, or--/
REA SOL NOD SOL	T	Alright, in other words, base 10./ Third./ (name)./
RES	S	The place value./
SOL	T	Can you tell me a little bit about what place value means as far as you are concerned?/
RES	S	Well, like if you had a number like 22, there

wouldn't be two two, it would be twenty - 2 because it's - the twenty is in the ten's place./

REA

T So you are saying then, where the digit is placed, has some effect on the value of that particular number. That's good./ Our next point./ Judy./

SOL
NOD SOL
RES

S 0. (mumbled) Our number system has 0./

REA

T Alright./ Why is this important? In terms of other number systems?/

SOL
RES

S Well, if we didn't have 0s, our number system would only go up to 10. We couldn't have anything higher than 10./

REA

T Alright, there are other reasons too why the 0 is important and we haven't really delved into it yet./ Alright, now, what's another principle that is important about our little number system?/

SOL

S Well, it has a principle of multiplication. That means you have to multiply to find out what the number is./

RES

SOL

T And finally, you might as well finish, Neil./

S There is a principle of addition too and you either got to multiply or add to find out what the number is./

RES

SOL

T Now, do you agree with Neil on this point? He said to multiply or add, is it multiply or add? What is it, Darrell?/

RES

S Multiply and add./

REA

T Yes, we really have to do both./ Now, lets think today of a numeral system which is based on another base other than base ten, just to see whether you can apply these principles of the numeral system to another base. The base that I have selected because I really think it is one of the easier ones, although it may not be for you, is one that is based on five (5) rather than 10. Or let us think of one that has as its base 5./ Now, I am going to listen while you tell me what

STR

SOL

this really means to you in terms of a numeral system. Can you apply the principles we have just discussed - a numeral system having base 5? What does this mean to you first of all, Dianne?/

RES
S Well, it has five digits. /

REA
T Alright,/ what would they be? /

SOL
RES
S 0, 1, 2, 3, 4./

REA
T Good./ Would a numeral system having base 5 still have place value?/

SOL
REA
S Yes./

STR
SOL
NOD SOL
T So we have to develop some sort of a place value chart./ Who can tell me, first of all, what the first column would be represented by? / Carol?/

RES
S 1. /

REA
T That will be a 1./ Now who can tell me what the second column would be represented by? Remember we are dealing with base 5./ (name)./

SOL
NOD SOL
RES
S 4? /

SOL
T How many of you think it would be 4? Not one of you?/

RES
S No. It's 5./

REA
T You say it would be 5. Alright,/ why do you think so? /

SOL
RES
S Well 1 times - times 5./

SOL
T Is that what we decided when we were talking about the base 10 number system?/

RES
S Well, in the base 10 number system you have to divide - multiply 1 by 10 and then in the, and then,/

REA
T So the second column is 10 times as big as the first in our base 10./ Now in the base 5 the second column has to be how many times as big as the first?/ Kim?/

SOL
NOD SOL

RES	S	5./
REA SOL NOD SOL	T	Alright./ Now, what will the third column be?/ Susan?/
RES	S	25./
SOL REA SOL NOD SOL	T	How many agree with Susan on that?/ Very good./ What about the next column?/ (name)./
RES	S	(mumbled)/
SOL NOD SOL	T	How many of you agree with that? Just one? Nobody?/ Darrell./
RES	S	125./
SOL	T	How did you get that?/
RES	S	Well, 125 is 5 times as big as 25./
REA SOL NOD SOL	T	Very good./ And we could continue on if we wanted to but now let us determine what the next column would be./ (name)./

UNIT IDENTIFICATION PRACTICE MATERIAL

The following transcription is to be used as practice material in learning to identify units. A key is given on pages 112 to 114.

T We'll continue where we left off with Miss B. with 125 over 121. What can you tell me about the 55, Joan?

S 11 goes into it 5 times.

T 11 goes into 55 5 times, these 2 factors being, Joan?

S 11 and 5.

T 11 and 5. If 55 over 121 can be reduced, what can you conclude? 55 may be broken down into 5 times 11. Assuming that 55 over 121 can be reduced, what can you conclude? Susan?

S Either 5 or 11 multiplied by the other (mumbled).

T Right. Either 5 or 11 must go into 121. Which of these do you see at a glance does not go? Kathy?

S 5.

T 5. How do you know 5 does not go into 121?

S It doesn't end in 5 or 0.

T That's right. It doesn't end in 5 or 0. Hence, 121 is not divisible by 5. But 121 is divisible by?

S 11.

T 11. In fact, 121 may be broken down into?

S 11 times 11.

T 11 times 11. So we have 5 times 11 over 11 times 11 and this reduces to? Eddy?

S 5/11

T 5/11. You cancel the 11 in the numerator and the 11 and you get 5/11. Here is another example. 91/156. Is there anyone who can see at a glance what the common factor is? You can?

S 7.

T 7. Well let's try 7. How many times does 7 go into 91?

S 13.

T 13 times, and how many times does 7 go into 917?

S It won't go.

T I won't go. Right. So it's not a common factor. Hence you can't reduce this by dividing numerator and denominator by 7. So I don't think you can see at a glance what the common factor is. So let's take a look at either the one or the other. 91. Is 91 divisable by any of these factors: 2,3,5? Doug?

S No.

T Fine. Why is it not divisible by 2?

S It isn't even.

T Therefore?

S If the last number.

T If the last number isn't even?

S It isn't a multiple of 2.

T Therefore it's odd. That's what I wanted from you. If it isn't even, it's odd. So that 91 is an odd number, and therefore, not divisible by 2. Why is 91 not divisible by 3? Jeff?

S Because it doesn't end in 1.

T Is that the reason why it's not divisible by 3?

S Yes.

T How do we decide without actually trying the division? How do we decide on how it is divisible by 3? Danette?

S Because 91 equals 10 and 3 won't go into 10 evenly.

T 3 won't go into 10 evenly, quite right. The sum of the digits is 10 and a 3 does not divide evenly into the sum of the digits. Hence 3 does not divide evenly into the number. So let's leave the 91 for the time being and take a look at 156. What can you tell me about 156, immediately? Peggy?

S Evenly divisible by 13.

T Evenly divisible by?

S 13.

T 13, good. (mumbled).

S Not exactly.

T Well, tell me exactly what you see looking at 156. You are quite right. It is divisible by 13, but unless you have another brain wave I don't think you would start dividing by 13. Laurie?

S (mumbled) It is an even number, so it would be a multiple of 2.

T It will be a multiple of 2. Quite right.

S It can also be divided by 3.

T It can also be divided by 3. Quite right. Why?

S (mumbled)

T 3 will go into 15 and the 6.

UNIT IDENTIFICATION KEY

The following transcription is the key to the practice material on pages 109 to 111.

STR SOL	T	We'll continue where we left off with Miss B. with 125 over 121./ What can you tell me about the 55, Joan?/
RES	S	11 goes into it 5 times./
REA SOL	T	11 goes into 55 5 times,/ these 2 factors being, Joan?/
RES	S	11 and 5./
REA SOL NOD SOL	T	11 and 5./ If 55 over 121 can be reduced, what can you conclude? 55 may be broken down into 5 times 11. Assuming that 55 over 121 can be reduced, what can you conclude?/ Susan?/
RES	S	Either 5 or 11 multiplied by the other (mumbled)./
REA SOL NOD SOL	T	Right. Either 5 or 11 must go into 121./ Which of these do you see at a glance does not go?/ Kathy?/
RES	S	5./
REA SOL	T	5./ How do you know 5 does not go into 121?/
RES	S	It doesn't end in 5 or 0./
REA SOL	T	That 's right. It doesn't end in 5 or 0. Hence, 121 is not divisible by 5./ But 121 is divisible by?/
RES	S	11./
REA SOL	T	11./ In fact, 121 may be broken down into?/
RES	S	11 times 11./
REA SOL NOD SOL	T	11 times 11./ So we have 5 times 11 over 11 times 11 and this reduces to?/ Eddy?/
RES	S	5/11./

REA T 5/11. You cancel the 11 in the numerator and the 11 and you get 5/11./ Here is
STR another example. 91/156./ Is there anyone who can see at a glance what the common
SOL factor is? You can?/

RES S 7./

REA T 7./ Well let's try 7. How many times does
SOL 7 go into 91?/

RES S 13./

REA T 13 times,/ and how many times does 7 go
SOL into 156?/

RES S It won't go./

T It won't go. Right. So it's not a common factor. Hence you can't reduce this by dividing numerator and denominator by 7.
REA So I don't think you can see at a glance what the common factor is./ So let's take
STR a look at either the one or the other. 91./ Is 91 divisable by any of these factors:
SOL 2, 3, 5?/ Doug?/
NOD SOL

RES S No./

REA
SOL T Fine./ Why is it not divisible by 2?/

RES S It isn't even./

SOL T Therefore?/

RES S If the last number./

SOL T If the last number isn't even?/

RES S It isn't a multiple of 2./

T Therefore it's odd. That's what I wanted from you. If it isn't even, it's odd. So
REA that 91 is an odd number, and therefore, not divisible by 2./ Why is 91 not divisible
SOL by 3?/ Jeff?/
NOD SOL

RES S Because it doesn't end in 1./

SOL RES	T	Is that the reason why it's not divisible by 3?/
	S	Yes./
SOL	T	How do we decide without actually trying the division? How do we decide on how it
NOD SOL		is divisible by 3?/ Anette?/
RES	S	Because 91 equals 10 and 3 won't go into 10 evenly./
	T	3 won't go into 10 evenly, quite right. The sum of the digits is 10 and a 3 does not divide evenly into the sum of the digits.
REA		Hence 3 does not divide evenly into the number./
STR		So let's leave the 91 for the time being and take a look at 156./
SOL NOD SOL		What can you tell me about 156, immediately?/ Peggy?/
RES	S	Evenly divisible by 13./
SOL	T	Evenly divisible by? /
RES	S	13./
REA	T	13, good./ (mumbled)
REA	S	Not exactly./
SOL	T	Well, tell me exactly what you see looking at 156. You are quite right. It is divisible by 13, but unless you have another brain wave I don't think you would start dividing by 13./
NOD SOL		Laurie?/
RES	S	(mumbled) It is an even number, so it would be a multiple of 2./
REA	T	It will be a multiple of 2. Quite right./
RES	S	It can also be divided by 3./
REA	T	It can also be divided by 3. Quite right./
SOL		Why?/
RES	S	(mumbled) /
REA	T	3 will go into 15 and the 6./

ILLUSTRATIONS OF THE CATEGORIES

The following units illustrate each of the categories of the instruments. Since no unit of preverbalization was found, hypothetical examples are given.

Units Illustrating the PREVERBALIZATION Category

SOL	T	Can someone give me a rational number of arithmetic which is equivalent to 2/3?/
RES	S	4/6./
SOL	T	Can someone give me some others?/
RES	S	8/12./
RES	S	16/24./
SOL	T	How do you know these are all equivalent to 2/3?/
RES	S	Well, they just are./

Units Illustrating the SYMMETRY Category

SOL	T	If I extended this and extended that, then this angle here would be equal to this one?/
RES	S	Yeah, and the same to the others./
REA	T	The same to the others./ Pardon? Go ahead./
NOD SOL RES	S	There's four triangles and these are equal because bisector, two bisectors for two pairs of equal triangles../ . . . . .
STR	T	So far, here again, you can't really prove that these two triangles are equal to that. As Bernie said, if this was an isosceles triangle or an equilateral triangle, which is a very special case. Therefore these two angles would be equal because this would be a right bisector, a right bisector here.

This is not necessarily true. So, Ruth, just to be able to do this, this one over here, as I say, you can prove that. Similarily, that this angle is less than this one, in exactly the same manner./

SOL S This angle here?/

RES T This angle over here, because now you can prove both parts. You've proven already, you've proven that this one is bigger than that remote interior./

Units Illustrating the SOLUTION VISUALIZATION Category

REA T A unit of area. Alright. Now we have a unit of area then./ Just exactly can you define
SOL or suggest what shape or size this might be?/

RES S A shape of a square./

REA T A shape of a square./

. . . .

REA T Alright./ Now, what about the n? The unknown here is the n in this particular equation and
SOL what will it tell us when we find what n is?/
NOD SOL Bruce?/

RES S Well, it tells us how many feet of ribbon each girl will get./

REA T Alright, and this is what we're after./

. . . . .

SOL T If we left it as two remainder what would happen to these two feet of ribbon?/

RES S They would just be wasted./

REA T And this isn't good economy, is it?/

. . . . . .

SOL T So when you end up, you put here, you want to have how many different piles of apples?/

RES S Twelve./

REA T You want to have twelve piles of apples.

That's right. So you know two things. You know your total and you know the total number of groups you want to end up with, and that's twelve./

Units Illustrating the ANALOGY Category

SOL T Can you tell me a little bit about what place value means as far as you are concerned?/

RES S Well, like if you had a number like twenty-two, there wouldn't be two two, it would be twenty-two, because it's - the twenty is in the tens place./

REA T So you are saying the, where the digit is placed, has some effect on the value of that particular number. That's good./

....

REA T Carry five of those? Right./ So you are
SOL keeping? Do you understand what Beth said?/
Look. Four and four in our numerals system, that's this many in terms of one thing - you
REA know if each of these fours were representing a lollipop for example - you actually put them down here - one, two, three, four, and one, two, three, four./

. . . .

REA T Five single pieces of paper left over./ Let's look back to the blackboard then. Five times three, Tony told us, was fifteen and Marshall told us we can make one bundle of ten and
STR have five left over. If we put it up here, in the spot where the tens are we'll put a one up here because you know we made one bundle and he has five left over./ If we
SOL put these groups of ten together the, the Monday through Friday groups, how many groups
NOD SOL of ten will we have then?/ Karen?/

RES S Five./

REA T Five groups of ten./

Units Illustrating the EXPERIMENTATION Category

REA SOL	T	Seven./ Well, let's try seven. How many times does seven go into ninety-one?/
RES	S	Thirteen./
REA SOL	T	Thirteen times./ And how many times does seven go into one hundred-and-fifty-six?/
		. . . .
SOL	T	Two and three-quarters what?/
RES	S	Segment./
REA	T	No./
RES	S	Squares./
		. . . .

Units Illustrating the GENERALIZATION Category

REA SOL	T	Quite right.?/ How about the terminal digit, the last digit?/
RES	S	It either ends in five or zero./
REA SOL	T	Good./ If a number ends in five or zero, what can you conclude?/
REA	S	That it's divisible by five./
STR SOL	T	Take 115, for example./ Does that end in five?/
		... .
REA SOL NOD SOL	T	Alright./ So the first term in our fraction is taken from where?/ Janet?/
RES	S	From the remainder.
REA SOL	T	Alright./ And the twelve is the?/
RES	S	Divisor./ Was this the case when we dealt with the ribbon? Remember, we had a fraction

SOL		here which was two over three. Was three our divisor? And was two our remainder?/
REA		Alright, so we have taken our remainder and made it the first term in our fraction./

Units Illustrating the INFORMAL PROOF Category

SOL	T	Thirty-seven goes into seventy-four how many times?/
RES	S	Twice./
REA NOD SOL	T	Twice. Alright then, this fraction reduces to two-thirds./ Yes Sid?/
REA	S	Given the sum of the numbers, it doesn't always work because if you add nine and one to ten and ninety-one isn't divisible by two./
REA	T	Nine, one./
		. . . .

Units Illustrating the MANIPULATION Category

SOL	T	Now what does that really mean? One - three is really one what?/
RES	S	One five./
		. . . .
SOL	T	When you are doing any type of problem like this, what are the things that you should consider before you really get down to writing that equation? Oh come now. What do you know about this problem?/
RES	S	Well, it's a dividing problem./
		. . . .
REA SOL	T	Alright, that's one thing about it. You know that you've got 129 pounds of apples./ What else do you know?/
RES	S	You know that there are twelve boys and you have to put them 129 into groups so each boy can get some apples./
		. . . .

SOL NOD SOL	T	What do we know in this particular problem?/ Gerald?/
RES	S	Twelve inches are one foot./
REA	T	Alright. So we have got one figure which is twelve, twelve being the number of inches in a foot./

. . . . . .

SOL NOD SOL	T	What else do we know already?/ Susan?/
RES	S	Three feet in one yard./

. . . . . .

SOL NOD SOL	T	What else do we know from this particular problem?/ Gerald?/
RES	S	Well, that there is forty-one inches./
REA	T	Alright, we are dealing with a piece of rope which is forty-one inches in total./

Units Illustrating the SIMILARITY category

REA STR SOL	T	Yes, we really have to do both. / Now, let's think today of a numeral system which is based on another base other than base ten, just to see whether you can apply these principles of the numeral system to another base. The base that I have selected, because I really think it is one of the easier ones, although it may not be for you, is one that is based on five rather than ten. Or let us think of one that has as its base 5./ Now, I am going to listen while you tell me what this really means to you in terms of a numeral system. Can you apply the principles we have just discussed to the numeral system having base five? What does this mean to you first of all, Dianne? /
RES	S	Well, it has five digits./
REA	T	Alright./

. . . . . .

SOL	T	Is that what we decided when we were talking about the base ten number system?/

RES	S	Well, in the base ten number system you have to divide - multiply one by 10 and then - /
REA	T	So the second column is ten times as big as the first in our base ten./ Now in the base five the second column has to be how many times as big as the first?/ Kim?/
SOL NOD SOL		
RES	S	Five./
REA	T	Alright./

. . . . . .

REA	T	Yes. There is no five as such, is there?/ Alright, now. What about the principle of multiplication? Would this still be in effect--and the principle of addition./
SOL		

. . . . . .

STR	T	Wanda, yesterday we had four unit segments in the one row in our rectangle. We said that we could have four to one or four unit segments in the one row./ Now this is equivalent to having how many segments in the three rows, if we had three rows of unit segments? Yes?/
SOL		
REA	S	I understand that./

. . . . . .

REA	T	Measure, O.K./ Now when we turn to the angle we don't use one inch for a unit of measure./ We use what?/
STR SOL		
RES	S	A degree./
REA	T	A degree, One degree would give us a unit of measure./

. . . . . .

REA	T	Size and shape./ Alright, now let's take the rectangle then and we want to develop a unit of measure for it. We have the interior and we have the (mumbled), and so our measure is going to be a measure of the interior and the simple closed curve./ Can we find a
STR		

SOL NOD SOL		way in which we may divide our closed region into some kind of a unit in order to find its measure? A segment, we had units of lengths, in an angle we had units which were degrees in measure. Now in our rectangle, can we find some kind of unit in order to get a measure of a rectangle? Now I am talking about the area, of course./ Carlo?/
RES	S	A unit of area./
REA	T	A unit of area. Alright. Now we have a unit of area then./

. . . .

SOL NOD SOL	T	Now using what you have done in the previous question, what can we do here in order to use this extra end here and make it more meaningful to this particular problem?/ Janet./
RES	S	Well, you take the five, which is the remainder and put it for the first term in a fraction and then you take the twelve, which is the divider, and make it the second term and you have a fraction of (mumbled)./
REA	T	Alright, this is part of a foot then./

Units Illustrating SKIPPING STEPS Category

REA STR SOL NOD SOL	T	Two-thirds, right./ Now here we concentrated on seventy-four. We could just as well have concentrated on the one hundred-and-eleven./ What can you tell me at a glance to be a factor of one hundred-and-eleven?/ Eddy?/
RES	S	Three./
REA SOL	T	Quite right./ Why?/

. . . . . .

STR SOL	T	Here is another example. Ninety-one over one hundred-and-fifty-six./ Is there anyone who can see at a glance what the common factor is? You can?/
RES	S	Seven./

REA T Seven./

. . . . . .

T Now let's continue on to page 243. You don't need to copy the fractions into your notebook but you're to fill in the missing terms. In some cases you need the numerator and in some cases you need the denominator./ Seven over twelve equals, - what over one hundred-and-fifty-six?/ Jeff?/

STR

SOL

NOD SOL

RES S Ninety-one./

SOL T Ninety-one over one hundred-and-fifty-six./ Jeff, how did you arrive at that ninety-one?/

. . . . . .

SOL T Now if you were saying that, giving this answer then, how many hours did Don work?, will you say then that he worked eight hours which is our quotient and twenty-five cents left over? Will that be a sufficient answer for this problem? What do we have to do here, Harold?/

RES S Well, I don't think it is very good. I think to get the proper answer, you would want eight and one-half hours./

SOL T How did you get that?/

. . . . . .